

Membrane quantum mechanics

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Abstract

We consider the multiple M2-branes wrapped on a compact Riemann surface and study the arising quantum mechanics by taking the limit where the size of the Riemann surface goes to zero. The IR quantum mechanical models resulting from the BLG-model and the ABJM-model compactified on a torus are $\mathcal{N} = 16$ and $\mathcal{N} = 12$ superconformal gauged quantum mechanics. After integrating out the auxiliary gauge fields we find $OSP(16|2)$ and $SU(1, 1|6)$ quantum mechanics from the reduced systems. The curved Riemann surface is taken as a holomorphic curve in a Calabi–Yau space to preserve supersymmetry and we present a prescription of the topological twisting. We find the $\mathcal{N} = 8$ superconformal gauged quantum mechanics that may describe the motion of two wrapped M2-branes in a K3 surface.

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1. Introduction

M2-brane appears to be a fundamental object in M-theory in the sense that it can be identified with the fundamental string after the compactification of M-theory to type IIA string theory [1]. In the past decade some progress has been made in finding the low-energy world-volume descriptions for multiple M2-branes. Inspired by the work in [2] and [3], Bagger, Lambert and

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Gustavsson discovered the three-dimensional $\mathcal{N} = 8$ superconformal Chern–Simons–matter theory, the so-called BLG-model [4–8]. Subsequently Aharony, Bergman, Jafferis and Maldacena constructed the three-dimensional $\mathcal{N} = 6$ superconformal Chern–Simons–matter theory, the so-called ABJM-model [9]. Since then, the BLG-model and the ABJM-model have been proposed as the low-energy effective world-volume theories of multiple planar M2-branes.

In this paper we study more general M2-branes wrapping a compact Riemann surface Σ_g of genus g . For $g \neq 1$ the world-volume of the M2-branes is curved and the Riemann surface has to be taken as a holomorphic curve in a Calabi–Yau manifold to preserve supersymmetry. The construction of such world-volume theories on the wrapped branes can be implemented as topologically twisted theories [10]. For the world-volume descriptions of wrapped M2-branes, we can take the further limit where the energy scale is much smaller than the inverse size of the Riemann surface. This implies that the Riemann surface shrinks to zero and thus the three-dimensional world-volume theories reduce to a one-dimensional field theories, i.e. quantum mechanics. The purpose of the present paper is to derive and study the emerging IR quantum mechanics by reducing the BLG-model and the ABJM-model.

It has been argued in [11] that there exist IR fixed points with AdS_2 factors in $d = 11$ supergravity solutions describing the M2-branes wrapping Σ_g which are gravity dual to superconformal quantum mechanics (SCQM). Quite interestingly we show that our low-energy effective quantum mechanics possesses a one-dimensional superconformal symmetry. Generally superconformal quantum mechanics is characterized by a supergroup that contains a one-dimensional conformal group $SL(2, \mathbb{R})$ and an R-symmetry group as factored bosonic subgroups. The first detailed analysis for a simple conformal quantum mechanical model, the so-called DFF-model is found in [12] and there has been a number of attempts to construct superconformal mechanics since the earliest work of [13,14]. One of the most powerful way to build such superconformal quantum mechanics is to resort to superspace and superfield formalism. However, it is unreasonable and unsuccessful for highly supersymmetric cases with $\mathcal{N} > 8$ supersymmetry because it is extremely difficult to pick up irreducible supermultiplets by imposing the appropriate constraints on the superfields [15]. Remarkably we find that such highly extended superconformal quantum mechanical models arise from the M2-branes wrapping a torus and that our reduced quantum mechanical actions agree with the predicted form for $\mathcal{N} > 4$ SCQM in [16–18].

This paper is organized as follows. In Section 2 we review the BLG-model [4–8] and the ABJM model [9]. In Section 3 we study the multiple M2-branes wrapped around a torus. From the BLG-model we find that the low-energy dynamics is described by $\mathcal{N} = 16$ superconformal gauged quantum mechanics. Furthermore, we show that $OSp(16|2)$ superconformal quantum mechanics appears from the reduced system after integrating out the auxiliary gauge field. Similarly from the ABJM-model $\mathcal{N} = 12$ superconformal gauged quantum mechanics makes an entrance at low-energy and we find the reduced quantum mechanics with $SU(1, 1|6)$ symmetry. In Section 4 we clarify the description for curved M2-branes wrapping a holomorphic curve in a Calabi–Yau manifold. We discuss the amount of preserved supersymmetries and establish a prescription for the topological twisting. In Section 5 we examine the two M2-branes wrapped on a Riemann surface of genus $g > 1$ embedded in a K3 surface in detail. Finally in Section 6 we conclude and discuss some directions for future research.

2. World-volume theories of M2-branes

2.1. BLG-model

The BLG-model is a three-dimensional $\mathcal{N} = 8$ superconformal Chern–Simons-matter theory proposed as a low energy world-volume theory of multiple M2-branes [4–8]. It is based on a 3-algebra \mathcal{A} , which is an N -dimensional vector space endowed with a trilinear skew-symmetric product $[A, B, C]$ satisfying

$$[A, B, [C, D, E]] = [[A, B, C], D, E] + [C, [A, B, D], E] + [C, D, [A, B, E]]. \quad (2.1)$$

This is called the fundamental identity and extends the Jacobi identity of Lie algebras to the 3-algebras. If we let T^a , $a = 1, \dots, N$ be a basis of the algebra, the 3-algebra is specified by the structure constants f^{abc}_d

$$[T^a, T^b, T^c] = f^{abc}_d T^d. \quad (2.2)$$

With the structure constant, the fundamental identity (2.1) can be expressed as

$$f^{abg}_h f^{cde}_g = f^{abc}_g f^{gde}_h + f^{abd}_g f^{cge}_h + f^{abe}_g f^{cdg}_h \quad (2.3)$$

Classification of the 3-algebras \mathcal{A} requires finding the solutions to the fundamental identity (2.3) for the structure constants f^{abc}_d .

In order to derive the equations of motion of the BLG-model from a Lagrangian description, a bi-invariant non-degenerate metric h^{ab} on the 3-algebra \mathcal{A} is needed. Bi-invariance requires the metric to satisfy $f^{abc}_e h^{ed} + f^{bcd}_e h^{ae} = 0$. This implies that the tensor $f^{abcd} \equiv f^{abc}_e h^{ed}$ is totally anti-symmetric. The metric h^{ab} arises by postulating a non-degenerate, bilinear scalar product $\text{Tr}(\cdot, \cdot)$ on the algebra \mathcal{A} :

$$h^{ab} = \text{Tr}(T^a, T^b). \quad (2.4)$$

The Lagrangian of the BLG-model is specified by the structure constant f^{abc}_d and the bi-invariant metric h^{ab} .

The field content of the BLG-model is eight real scalar fields $X^I = X^I_a T^a$, $I = 1, \dots, 8$, fermionic fields $\Psi_{\dot{A}a} = \Psi_{\dot{A}a} T^a$, $\dot{A} = 1, \dots, 8$, and non-propagating gauge fields $A_{\mu ab}$, $\mu = 0, 1, 2$. The bosonic scalar fields X^I and the fermionic fields $\Psi_{\dot{A}}$ are $\mathbf{8}_v$ and $\mathbf{8}_c$ of an $SO(8)$ R-symmetry respectively. Also they are the fundamental representations of the 3-algebra. Gauge fields $A_{\mu ab}$ are the 3-algebra valued world-volume vector fields. They are anti-symmetric under two indices a, b of the 3-algebra $A_{\mu ab} = -A_{\mu ba}$.

$\Psi_{\dot{A}a}$ is defined as an $SO(1, 10)$ Majorana fermion and its conjugate is given by

$$\bar{\Psi} := \Psi^T \mathcal{C} \quad (2.5)$$

where \mathcal{C} is the $SO(1, 10)$ charge conjugation matrix satisfying

$$\mathcal{C}^T = -\mathcal{C}, \quad \mathcal{C} \Gamma^M \mathcal{C}^{-1} = -(\Gamma^M)^T. \quad (2.6)$$

Gamma matrix Γ^M is the representation of the $SO(1, 10)$ Clifford algebra

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}, \quad \Gamma^{10} := \Gamma^{0\dots 9} \quad (2.7)$$

where $\eta^{MN} = \text{diag}(-1, +1, +1, \dots, +1)$. Γ^M can be decomposed as

$$\begin{cases} \Gamma^\mu = \gamma^\mu \otimes \tilde{\Gamma}^9, & \mu = 0, 1, 2 \\ \Gamma^I = \mathbb{I}_2 \otimes \tilde{\Gamma}^{I-2}, & I = 3, \dots, 10 \end{cases} \quad (2.8)$$

where

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3 \quad (2.9)$$

and $\tilde{\Gamma}^I$ is the $SO(8)$ 16×16 gamma matrix whose chirality matrix is defined as $\tilde{\Gamma}^9 := \tilde{\Gamma}^{1\dots 8}$. Correspondingly the charge conjugation matrix can be expanded as

$$\mathcal{C} = \gamma^0 \otimes \tilde{\mathcal{C}} \quad (2.10)$$

where $\tilde{\mathcal{C}}$ denotes the $SO(8)$ charge conjugation matrix satisfying

$$\tilde{\mathcal{C}}^T = \tilde{\mathcal{C}}, \quad \tilde{\mathcal{C}} \tilde{\Gamma}^I \tilde{\mathcal{C}}^{-1} = -(\tilde{\Gamma}^I)^T. \quad (2.11)$$

The fermionic field Ψ is the real $\frac{1}{2} \cdot 2^{\lfloor \frac{11}{2} \rfloor} = 32$ -component Majorana spinor of eleven-dimensional space-time obeying the chirality condition

$$\Gamma^{012} \Psi = -\Psi. \quad (2.12)$$

Although at this stage Ψ has sixteen independent real components, they are reduced to eight when we treat it on-shell. From (2.8) it follows that

$$\Gamma^{012} = \Gamma^{34\dots 10} = \mathbb{I}_2 \otimes \tilde{\Gamma}^9 \quad (2.13)$$

and

$$\Gamma^{34\dots 10} \Psi = -\Psi. \quad (2.14)$$

This implies that Ψ is the conjugate spinor representation $\mathbf{8}_c$ of the $SO(8)_R$ R-symmetry group.

The Lagrangian of the BLG-model is

$$\begin{aligned} \mathcal{L}_{\text{BLG}} = & -\frac{1}{2} D^\mu X^{Ia} D_\mu X_a^I + \frac{i}{2} \bar{\Psi}_{\dot{A}}^a \Gamma_{\dot{A}\dot{B}}^\mu D_\mu \Psi_{\dot{B}a} \\ & + \frac{i}{4} \bar{\Psi}_{\dot{A}b} \Gamma_{\dot{A}\dot{B}}^{IJ} X_c^I X_d^J \Psi_{\dot{B}a} f^{abcd} - V(X) + \mathcal{L}_{\text{TCS}} \end{aligned} \quad (2.15)$$

where

$$V(X) = \frac{1}{12} f^{abcd} f^{efg}{}_d X_a^I X_b^J X_c^K X_e^I X_f^J X_g^K \quad (2.16)$$

$$\mathcal{L}_{\text{TCS}} = \frac{1}{2} \epsilon^{\mu\nu\lambda} \left(f^{abcd} A_{\mu ab} \partial_\nu A_{\lambda cd} + \frac{2}{3} f^{cda}{}_g f^{efgb} A_{\mu ab} A_{\nu cd} A_{\lambda ef} \right). \quad (2.17)$$

The covariant derivative is defined as

$$D_\mu X_a := \partial_\mu X_a - \tilde{A}_{\mu a}^b X_b \quad (2.18)$$

where $\tilde{A}_{\mu b}^a := f^{cda}{}_b A_{\mu cd}$. Although the kinetic term of the gauge fields is similar to the conventional Chern–Simons term, it is twisted by the structure constant of the 3-algebra. The gauge fields are non-propagating since they have at most first order derivative terms.

The supersymmetry transformations of the BLG-model are

$$\delta X_a^I = i\bar{\epsilon}_A \Gamma_{\dot{A}\dot{B}}^I \Psi_{\dot{B}a} \quad (2.19)$$

$$\delta \Psi_{\dot{A}a} = D_\mu X_a^I \Gamma_{\dot{A}\dot{B}}^I \epsilon_B - \frac{1}{6} X_b^I X_c^J X_d^K f^{bcd} \Gamma_{\dot{A}\dot{B}}^{IJK} \epsilon_B \quad (2.20)$$

$$\delta \tilde{A}_{\mu a}^b = i\bar{\epsilon}_A \Gamma_\mu \Gamma_{\dot{A}\dot{B}}^I X_c^I \Psi_{\dot{B}d} f^{cdb}{}_a \quad (2.21)$$

where ϵ is the unbroken supersymmetry parameter obeying the chirality condition

$$\Gamma^{012} \epsilon = \Gamma^{34 \dots 10} \epsilon = \epsilon. \quad (2.22)$$

This means that ϵ transforms as the spinor representation $\mathbf{2}$ of the $SL(2, \mathbb{R})$ and transforms as the spinor representation $\mathbf{8}_s$ of the $SO(8)_R$ R-symmetry. The action (2.15) is invariant under the supersymmetry transformations (2.19)–(2.21) up to a surface term.

If we assume that (i) the metric h^{ab} of the 3-algebra \mathcal{A} is positive-definite so that the kinetic term and the potential term are all positive, and that (ii) the dimension N of the 3-algebra \mathcal{A} is finite, then non-trivial 3-algebra \mathcal{A} is uniquely determined as [19,20]

$$f^{abcd} = \frac{2\pi}{k} \epsilon^{abcd} =: f \epsilon^{abcd} \quad (2.23)$$

$$h^{ab} = \delta^{ab} \quad (2.24)$$

where the gauge indices a, b, \dots run from 1 to 4 and k is the integer valued Chern–Simons level. This is called the \mathcal{A}_4 algebra. For the \mathcal{A}_4 algebra one can realize two gauge groups, $SO(4) = SU(2) \times SU(2)/\mathbb{Z}_2$ and $Spin(4) = SU(2) \times SU(2)$ [21]. The moduli space for \mathcal{A}_4 BLG-model with level k is identified with [21]

$$\mathcal{M}_k = \begin{cases} \frac{\mathbb{R}^8 \times \mathbb{R}^8}{D_{2k}} & \text{for } SO(4) \\ \frac{\mathbb{R}^8 \times \mathbb{R}^8}{D_{4k}} & \text{for } Spin(4). \end{cases} \quad (2.25)$$

The limitation on the rank of the gauge algebra may only allow the BLG-model to describe two M2-branes in analogy with D-branes.¹

2.2. ABJM-model

The ABJM-model is a three-dimensional $\mathcal{N} = 6$ superconformal $U(N)_k \times \hat{U}(N)_{-k}$ Chern–Simons-matter theory proposed as a generalization of the BLG-model in that it may describe the dynamics of an arbitrary number of coincident M2-branes [9]. The theory has manifestly only $\mathcal{N} = 6$ supersymmetry and the corresponding $SU(4)_R$ R-symmetry at the classical level. It has been discussed that [9,22,23] at $k = 1$ and $k = 2$ these symmetries are enhanced to $\mathcal{N} = 8$ supersymmetry and $SO(8)_R$ R-symmetry as a quantum effect.

The theory contains four complex scalar fields Y^A , four complex spinors ψ_A and two different types of gauge fields A_μ and \hat{A}_μ . Here the upper and lower indices $A, B, \dots = 1, 2, 3, 4$ denote $\mathbf{4}$ and $\bar{\mathbf{4}}$ of the $SU(4)_R$ respectively. The matter fields are $N \times N$ matrices so that Y^A and ψ_A transform as (N, \bar{N}) bi-fundamental representations of $U(N)_k \times \hat{U}(N)_{-k}$ gauge group, while

¹ In this paper we will focus on the \mathcal{A}_4 algebra, however, the Nambu–Poisson 3-algebra and the Lorentzian 3-algebra have been proposed as the escapes from the restriction by relaxing the condition on dimensionality and the requirement of a positive-definiteness metric respectively.

Y_A^\dagger and $\psi^{\dagger A}$ do as (\bar{N}, N) . A_μ is a Chern–Simons $U(N)$ gauge field of level $+k$ and \hat{A}_μ is that of level $-k$. Also in the theory there is a $U(1)_B$ flavor symmetry and the corresponding baryonic charges are assigned $+1$ for bi-fundamental fields, -1 for anti-bi-fundamental fields and 0 for gauge fields.

The Lagrangian of the ABJM-model is given by [24]

$$\begin{aligned} \mathcal{L}_{\text{ABJM}} = & -\text{Tr}(D_\mu Y_A^\dagger D^\mu Y^A) - i \text{Tr}(\psi^{\dagger A} \gamma^\mu D_\mu \psi_A) - V_{\text{ferm}} - V_{\text{bos}} \\ & + \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \text{Tr} \left[A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right] \end{aligned} \quad (2.26)$$

where

$$\begin{aligned} V_{\text{ferm}} = & -\frac{2\pi i}{k} \text{Tr}(Y_A^\dagger Y^A \psi^{\dagger B} \psi_B - \psi^{\dagger B} Y^A Y_A^\dagger \psi_B - 2Y_A^\dagger Y^B \psi^{\dagger A} \psi_B + 2Y^A Y_B^\dagger \psi_A \psi^{\dagger B} \\ & - \epsilon^{ABCD} Y_A^\dagger \psi_B Y_C^\dagger \psi_D + \epsilon_{ABCD} Y^A \psi^{\dagger B} Y^C \psi^{\dagger D}) \end{aligned} \quad (2.27)$$

$$\begin{aligned} V_{\text{bos}} = & -\frac{4\pi^2}{3k^2} \text{Tr}(Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + Y_A^\dagger Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C \\ & + 4Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger - 6Y^A Y_B^\dagger Y^B Y_A^\dagger Y^C Y_C^\dagger). \end{aligned} \quad (2.28)$$

Here we use the Dirac matrix $(\gamma^\mu)_\alpha{}^\beta = (i\sigma_2, \sigma_1, \sigma_3)$. The spinor indices are raised, $\theta^\alpha = \epsilon^{\alpha\beta} \theta_\beta$, and lowered, $\theta_\alpha = \epsilon_{\alpha\beta} \theta^\beta$ with $\epsilon^{12} = -\epsilon_{12} = 1$. Note that this makes the Dirac matrix $\gamma_{\alpha\beta}^\mu := (\gamma^\mu)_\alpha{}^\gamma \epsilon_{\gamma\beta} = (-\mathbb{I}_2, -\sigma_3, \sigma_1)$ symmetric and guarantees the Hermiticity of the fermionic kinetic term. The covariant derivatives are defined by

$$\begin{aligned} D_\mu Y^A &= \partial_\mu Y^A + iA_\mu Y^A - iY^A \hat{A}_\mu, & D_\mu \psi_A &= \partial_\mu \psi_A + iA_\mu \psi_A - i\psi_A \hat{A}_\mu \\ D_\mu Y_A^\dagger &= \partial_\mu Y_A^\dagger - iA_\mu Y_A^\dagger + iY_A^\dagger \hat{A}_\mu, & D_\mu \psi^{\dagger A} &= \partial_\mu \psi^{\dagger A} - iA_\mu \psi^{\dagger A} + i\psi^{\dagger A} \hat{A}_\mu. \end{aligned} \quad (2.29)$$

The supersymmetry transformation laws are

$$\delta Y^A = i\omega^{AB} \psi_B \quad (2.30)$$

$$\delta Y_A^\dagger = i\psi^{\dagger B} \omega_{AB} \quad (2.31)$$

$$\delta \psi_A = -\gamma^\mu \omega_{AB} D_\mu Y^B + \frac{2\pi}{k} [-\omega_{AB} (Y^C Y_C^\dagger Y^B - Y^B Y_C^\dagger Y^C) + 2\omega_{CD} Y^C Y_A^\dagger Y^D] \quad (2.32)$$

$$\delta \psi^{\dagger A} = D_\mu Y_B^\dagger \omega^{AB} \gamma^\mu + \frac{2\pi}{k} [-(Y_B Y^C Y_C^\dagger - Y_C^\dagger Y^C Y_B^\dagger) \omega^{AB} + 2Y_D^\dagger Y^A Y_C^\dagger \omega^{CD}] \quad (2.33)$$

$$\delta A_\mu = \frac{\pi}{k} (-Y^A \psi^{\dagger B} \gamma_\mu \omega_{AB} + \omega^{AB} \gamma_\mu \psi_A Y_B^\dagger) \quad (2.34)$$

$$\delta \hat{A}_\mu = \frac{\pi}{k} (-\psi^{\dagger A} Y^B \gamma_\mu \omega_{AB} + \omega^{AB} \gamma_\mu Y_A^\dagger \psi_B). \quad (2.35)$$

The parameter ω_{AB} is defined by

$$\omega_{AB} := \epsilon_i (\Gamma^i)_{AB}, \quad \omega^{AB} := \epsilon_i (\Gamma^{i*})^{AB} \quad (2.36)$$

where the $SL(2, \mathbb{R})$ spinor ϵ^i , $i = 1, \dots, 6$, transforms as the representation **6** under the $SU(4)_R$ and Γ^i is the six-dimensional 4×4 matrix satisfying

$$(\Gamma^i)_{AB} = -(\Gamma^i)_{BA} \quad (2.37)$$

$$\frac{1}{2}\epsilon^{ABCD}(\Gamma^i)_{CD} = -(\Gamma^{i\dagger})^{AB} = (\Gamma^{i*})^{AB} \quad (2.38)$$

$$\{\Gamma^i, \Gamma^j\} = 2\delta_{ij}. \quad (2.39)$$

Note that the supersymmetry parameter ω_{AB} obeys

$$\omega^{AB} = \omega_{AB}^* = \frac{1}{2}\epsilon^{ABCD}\omega_{CD}. \quad (2.40)$$

The moduli space of the $U(N)_k \times \hat{U}(N)_{-k}$ ABJM-model is [9]

$$\mathcal{M}_{N,k} = \frac{(\mathbb{C}^4/\mathbb{Z}_k)^N}{S_N} = \text{Sym}^N(\mathbb{C}^4/\mathbb{Z}_k). \quad (2.41)$$

This can be identified with the moduli space of N indistinguishable M2-branes moving in $\mathbb{C}^4/\mathbb{Z}_k$ transverse space. Therefore the ABJM-model is expected to describe the low-energy world-volume theory of N coincident M2-branes probing an orbifold $\mathbb{C}^4/\mathbb{Z}_k$. The four complex scalar fields Y^A represent the positions of the membranes in \mathbb{C}^4 .

In [21] it has been discussed that if N and k are co-prime, then the vacuum moduli space of the $U(N)_k \times \hat{U}(N)_{-k}$ theory is equivalent to that of the $SU(N) \times SU(N)/\mathbb{Z}_N$ theory. Consequently there are conjectural dualities between the ABJM theory and the BLG theory

$$U(2)_1 \times \hat{U}(2)_{-1} \text{ ABJM theory} \quad \Leftrightarrow \quad SO(4) \text{ BLG theory with } k=1 \quad (2.42)$$

$$U(2)_2 \times \hat{U}(2)_{-2} \text{ ABJM theory} \quad \Leftrightarrow \quad Spin(4) \text{ BLG theory with } k=2. \quad (2.43)$$

These proposed dualities have been tested by the computations of the superconformal indices [25]. Hence we may regard the $SO(4)$ BLG-model with $k=1$ as the world-volume theory of two planar M2-branes propagating in a flat space.

3. SCQM from flat M2-branes

3.1. $\mathcal{N} = 16$ superconformal mechanics

3.1.1. Derivation of quantum mechanics

We begin our discussion with the BLG-model in the case where the membranes wrap a torus T^2 and propagate in a transverse space with an $SO(8)$ holonomy group. In this case the world-volume theory of M2-branes is given by the action (2.15) defined on $M_3 = \mathbb{R} \times T^2$.

In general a torus can be characterized by two periods in the complex plane. Such periods are defined as the integration of a holomorphic differential ω along two canonical homology basis a , b of a torus. Let us define the periods by

$$\int_a \omega = 1, \quad \int_b \omega = \tau \quad (3.1)$$

where τ is the moduli of the torus and it should not be real.

In the following we want to consider the limit in which T^2 has vanishingly small size and derive the low-energy effective one-dimensional theory on \mathbb{R} . In order to obtain such a theory we need to determine the configurations with the lowest energy. Since we are now considering supersymmetric theories, the conditions are expressed as the BPS equations. As we are interested in

bosonic BPS configurations, we require that the background values of the fermionic fields vanish. Then the bosonic fields are automatically invariant under their supersymmetry transformations. Therefore the BPS equations correspond to the vanishing of the supersymmetry transformations (2.20) for fermionic fields. Also we discard the terms which include the covariant derivatives with respect to time because we are now interested in the low energy dynamics as a fluctuation around gauge invariant static configurations. Then one finds the BPS equations

$$D_z X_a^I = 0, \quad D_{\bar{z}} X_a^I = 0 \quad (3.2)$$

$$[X^I, X^J, X^K] = 0. \quad (3.3)$$

To go further we consider the $SO(4)$ BLG-model that may describe two M2-branes. In this case the Higgs fields transform as fundamental representations of the $SO(4)$ gauge group and we assume that these Higgs fields have non-zero values. Then the generic solution to (3.3) is given by $X_a^I = (X_1^I, X_2^I, 0, 0)^T$. For these solutions, the remaining BPS equations (3.2) reduce to

$$\partial_z X_1^I + \tilde{A}_{z2}^1 X_2^I = 0, \quad \partial_z X_2^I - \tilde{A}_{z2}^1 X_1^I = 0 \quad (3.4)$$

$$\tilde{A}_{z3}^1 X_1^I + \tilde{A}_{z3}^2 X_2^I = 0, \quad \tilde{A}_{z4}^1 X_1^I + \tilde{A}_{z4}^2 X_2^I = 0 \quad (3.5)$$

and their complex conjugates. First of all, Eqs. (3.4) tell us that the sum of the squares $(X_1^I)^2 + (X_2^I)^2$ for $I = 1, \dots, 8$ is independent of the locus of the Riemann surface. Thus we can write

$$X_1^{I+2} + i X_2^{I+2} = r^I e^{i(\theta^I + \varphi(z, \bar{z}))} \quad (3.6)$$

where $r^I, \theta^I \in \mathbb{R}$ are constant on the torus and represent the configuration of the two membranes in the I -th direction while $\varphi(z, \bar{z})$ may depend on z and \bar{z} . Furthermore, Eqs. (3.4) enable us to write $\tilde{A}_{z2}^1 = \partial_z \varphi$. The second set of equations (3.5) forces us to turn off four of six gauge fields; $\tilde{A}_{z3}^1 = \tilde{A}_{z3}^2 = \tilde{A}_{z4}^1 = \tilde{A}_{z4}^2 = 0$. These components of the gauge field become massive by the Higgs mechanism. Note that the above set of solutions automatically satisfies the integrability condition for (3.2) because the gauge field \tilde{A}_{z2}^1 is flat.

One can find further restrictions by noting that the flat gauge fields \tilde{A}_{z2}^1 on a torus have specific expressions. Cutting a torus along the canonical basis a and b , the sections of a flat bundle are described by their transition functions, i.e. constant phases around a and b . Thus they can be completely classified by their twists $e^{2\pi i \xi}, e^{-2\pi i \zeta}$ on the homology along cycles a, b where ξ and ζ are real parameters. This space is the torus \mathbb{C}/L_τ where L_τ is the lattice generated by $\mathbb{Z} + \tau\mathbb{Z}$. It is referred to as the Jacobi variety of T^2 denoted by $\text{Jac}(T^2)$. The twists on the homology can be described as a point on the Jacobi variety. Hence the flat gauge field can be expressed in the form [26]

$$\tilde{A}_{z2}^1 = -2\pi \frac{\Theta}{\tau - \bar{\tau}} \omega, \quad \tilde{A}_{\bar{z}2}^1 = 2\pi \frac{\bar{\Theta}}{\tau - \bar{\tau}} \bar{\omega} \quad (3.7)$$

where $\Theta := \zeta + \bar{\tau}\xi$ is the complex parameter representing the twists on the homology along two cycles. Subsequently we can write

$$\varphi(z, \bar{z}) = 2\pi \frac{\bar{\Theta}}{\tau - \bar{\tau}} \bar{z} - 2\pi \frac{\Theta}{\tau - \bar{\tau}} z. \quad (3.8)$$

Recalling that the angular variable $\varphi(z, \bar{z})$ in the $X_1^I X_2^I$ -plane characterizes the ratio of two bosonic degrees of freedom for the two membranes, it must take same values modulo $2\pi\mathbb{Z}$ under the shifts $z \rightarrow z + 1$ and $z \rightarrow z + \tau$ around two cycles. This implies that both the coordinates

ξ and ζ can only have integer values, namely \tilde{A}_{z2}^1 and $\tilde{A}_{\bar{z}2}^1$ are quantized. Therefore the generic BPS solutions are given by

$$X^{I+2} = \begin{pmatrix} X_A^I \\ X_B^I \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta^I + \varphi(z, \bar{z})) \\ \sin(\theta^I + \varphi(z, \bar{z})) \\ 0 \\ 0 \end{pmatrix} r^I$$

$$\tilde{A}_z = \begin{pmatrix} 0 & -2\pi \frac{\Theta}{\tau - \bar{\tau}} \omega_z & 0 & 0 \\ 2\pi \frac{\Theta}{\tau - \bar{\tau}} \omega_z & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{A}_{z4}^3(z, \bar{z}) \\ 0 & 0 & -\tilde{A}_{z4}^3(z, \bar{z}) & 0 \end{pmatrix}. \quad (3.9)$$

Here \tilde{A}_{z4}^3 and $\tilde{A}_{\bar{z}4}^3$ are the Abelian gauge fields associated with the preserved $U(1)$ symmetry and have no constraints from the BPS conditions. Taking into account the bosonic configurations (3.9) and the supersymmetry transformations (2.19), we introduce fermionic partners

$$\psi_{\pm} = \begin{pmatrix} \Psi_{\pm A} \\ \Psi_{\pm B} \\ 0 \\ 0 \end{pmatrix}, \quad \bar{\psi}^{\pm} = \begin{pmatrix} \bar{\Psi}^{\pm}_A \\ \bar{\Psi}^{\pm}_B \\ 0 \\ 0 \end{pmatrix} \quad (3.10)$$

where $\bar{\Psi}$ is the conjugate spinor defined by $\bar{\Psi} := \Psi^T \tilde{C}$ in terms of the $SO(8)$ charge conjugation matrix \tilde{C} . Ψ_+^a and $\bar{\Psi}^{+a}$ are the $SO(2)_E$ spinors with the positive chiralities while Ψ_-^a and $\bar{\Psi}^{-a}$ carry the negative ones. Both of them transform as $\mathbf{8}_c$ of the $SO(8)_R$.

Given the above static BPS configurations (3.9) and (3.10), we now wish to consider the evolution of time and compactify the system on T^2 . Substitution of the configurations (3.9) and (3.10) into the action (2.15) yields

$$S = \int_{\mathbb{R}} dt \int_{T^2} d^2z \left[\frac{1}{2} D_0 X^{Ia} D_0 X_a^I - \frac{i}{2} \bar{\Psi}^{aa} D_0 \Psi_{\alpha a} \right. \\ \left. - \frac{k}{2\pi} \tilde{A}_{02}^1 \tilde{F}_{z\bar{z}4}^3 - \frac{k}{4\pi} (\tilde{A}_{z2}^1 \dot{\tilde{A}}_{\bar{z}4}^3 - \tilde{A}_{\bar{z}2}^1 \dot{\tilde{A}}_{z4}^3) \right] \quad (3.11)$$

where the Greek letters $\alpha = +, -$ denote the $SO(2)_E$ spinor indices. The terms in the first line of the action (3.11) come from the kinetic terms of the BLG action while those in the second correspond to the twisted topological Chern–Simons terms.

Firstly since the gauge fields \tilde{A}_{z2}^1 and $\tilde{A}_{\bar{z}2}^1$ are quantized and their time derivatives do not appear in the action, these fields are just auxiliary fields. Exploiting the equations of motion they can be excluded and we find the constraints $\dot{\tilde{A}}_{z4}^3 = \dot{\tilde{A}}_{\bar{z}4}^3 = 0$. Hence the corresponding field strength $\tilde{F}_{z\bar{z}4}^3$ has no time dependence. In order to dimensionally reduce the theory on the torus, we rescale the fields as

$$X^{I'} = R^2 X^I, \quad \Psi'_{\alpha a} = R^2 \Psi_{\alpha a}, \quad \bar{\Psi}^{\alpha a'} = R^2 \bar{\Psi}^{\alpha a} \quad (3.12)$$

where R is the circumference of the torus. Note that they get the canonical dimensions in the reduced theory; the bosonic variable $X^{I'}$ has mass dimension $-1/2$ and the fermionic variable Ψ' acquires mass dimension 0.

Performing the integration on the torus by means of the Kaluza–Klein ansatz for \tilde{A}_0^{12} and dropping the primes on the fields, one finds the effective action

$$S = \int_{\mathbb{R}} dt \left[\frac{1}{2} D_0 X^{Ia} D_0 X_a^I - \frac{i}{2} \bar{\Psi}^{\alpha a} D_0 \Psi_{\alpha a} - k C_1(E) \tilde{A}_{02}^1 \right]. \quad (3.13)$$

Here

$$C_1(E) = \int_{T^2} c_1(E) := \frac{1}{2\pi} \int_{T^2} d^2 z \tilde{F}_{z\bar{z}4}^3 \quad (3.14)$$

is the Chern number resulting from the integration of the first Chern class $c_1(E)$ of the $U(1)$ principal bundle $E \rightarrow T^2$ over the torus, which is associated with the preserved $U(1)$ gauge field \tilde{A}_{z4}^3 .

The action (3.13) is invariant under the one-dimensional conformal transformations

$$\delta t = f(t) = a + bt + ct^2, \quad \delta \partial_0 = -\dot{f} \partial_0 \quad (3.15)$$

$$\delta X_a^I = \frac{1}{2} \dot{f} X_a^I, \quad \delta \tilde{A}_{02}^1 = -\dot{f} \tilde{A}_{02}^1 \quad (3.16)$$

$$\delta \Psi_{\alpha a} = 0, \quad \delta \bar{\Psi}^{\alpha a} = 0 \quad (3.17)$$

where $f(t)$ is a quadratic function of time with real infinitesimal parameters a , b and c .

Besides, the action (3.13) is invariant under the $\mathcal{N} = 16$ supersymmetry transformations

$$\delta X_a^I = i\bar{\epsilon}^+ \tilde{F}^I \Psi_{-a} - i\bar{\epsilon}_- \tilde{F}^I \Psi_{+a}, \quad \delta \tilde{A}_{02}^1 = 0 \quad (3.18)$$

$$\delta \Psi_{+a} = -D_0 X_a^I \tilde{F}^I \epsilon_-, \quad \delta \Psi_{-a} = D_0 X_a^I \tilde{F}^I \epsilon_+. \quad (3.19)$$

Therefore the resulting effective theory (3.13) takes the form of $\mathcal{N} = 16$ superconformal gauged quantum mechanics with a Fayet–Iliopoulos (FI) term.

3.1.2. Reduced system with inverse-square interaction

Since the gauged mechanical action (3.13) is quadratic in the $U(1)$ gauge field \tilde{A}_{02}^1 and does not involve the time derivative of it, \tilde{A}_{02}^1 is identified with an auxiliary field and has no contribution to the Hamiltonian. Hence the Hamiltonian is invariant under the action of the corresponding $U(1)$ gauge group on the phase space \mathcal{M} . This means that the corresponding moment map $\mu: \mathcal{M} \rightarrow \mathfrak{u}(1)^*$ is the integral of motion [27] and one can reduce the given phase space \mathcal{M} to a smaller one $\mathcal{M}_c = \mu^{-1}(c)$ with fewer degrees of freedom by fixing the inverse of the moment map at a point $c \in \mathfrak{u}(1)^*$.²

In fact it is known that one-dimensional gauged matrix models give rise to the alternative descriptions of the Calogero model and its generalizations as the reduced systems [28–30]. In order to obtain our reduced system, we shall eliminate the auxiliary field \tilde{A}_{02}^1 in two steps; first we choose a specific gauge and then impose the Gauss law constraint to ensure the consistency of the gauge fixing. Let us choose the temporal gauge $\tilde{A}_0 = 0$. Together with the solutions

$$\tilde{A}_{02}^1 = \frac{k C_1(E) + \sum_I (r^I)^2 \dot{\theta}^I + i \bar{\Psi}_A^\alpha \Psi_{\alpha B}}{\sum_I (r^I)^2} \quad (3.20)$$

$$\tilde{A}_{03}^1 = \tilde{A}_{04}^1 = \tilde{A}_{03}^2 = \tilde{A}_{04}^2 = 0 \quad (3.21)$$

² The components of the moment map form a system being in involution since the gauge group is Abelian. So we do not need to divide by the non-trivial coadjoint isotropy subgroup to obtain the reduced phase space.

to the equations of motion for \tilde{A}_0 , we can read off the Gauss law constraint

$$\phi_0 := kC_1(E) + \sum_I (r^I)^2 \dot{\theta}^I + i\bar{\Psi}_A^\alpha \Psi_{\alpha B} = 0. \quad (3.22)$$

This equation is the moment map condition. To see the physical meaning of this constraint, we observe that $(r^I)^2 \dot{\theta}^I$ represents the “angular momentum”, the $SO(2)$ -charge corresponding to the rotation in the $X_1^I X_2^I$ -plane while the fermionic bilinear term $i\bar{\Psi}_A^\alpha \Psi_{\alpha B}$ produces the charge of the $SO(2)$ rotational group of the two types of fermionic variables Ψ_A and Ψ_B . Accordingly the equation (3.22) says that the total $SO(2)$ charge which rotates the internal degrees of freedom for the two M2-branes is fixed by the Chern–Simons level k and the Chern number $C_1(E)$.

With the constraint function ϕ_0 , one can write a new Lagrangian by adding $\lambda\phi_0$ where λ is the Lagrange multiplier. The resulting action is

$$S = \int_{\mathbb{R}} dt \left[\frac{1}{2} \sum_I (\dot{r}^I)^2 + \frac{1}{2} \sum_I (r^I \dot{\theta}^I)^2 - \frac{i}{2} \bar{\Psi}^{\alpha a} \dot{\Psi}_{\alpha a} + \lambda \left(kC_1(E) + \sum_I (r^I)^2 \dot{\theta}^I + i\bar{\Psi}_A^\alpha \Psi_{\alpha B} \right) \right]. \quad (3.23)$$

The absence of the variables θ^I 's in the action (3.23) immediately implies that they are cyclic coordinates and their canonical momenta $p_{\theta^I} = (r^I)^2 \dot{\theta}^I$ are just the integrals of motion.

In what follows we reduce the theory through the Routh reduction (see for example [31]). It should be emphasized that although the Hamiltonian reduction³ may be a more familiar method to eliminate variables with symmetries, in our case the Routh reduction provides us a simpler and more direct path to obtain the explicit action or the Lagrangian of the superconformal mechanics with $\mathcal{N} > 8$ supersymmetry unavailable so far. It is possible to eliminate cyclic coordinates from the Lagrangian by constructing a new Lagrangian, the so-called Routhian [32]. The Routhian is a hybrid between the Lagrangian and the Hamiltonian, defined by performing a Legendre transformation on the cyclic coordinates

$$R(r^I, \dot{r}^I, h^I, \Psi) := L(r^I, \dot{r}^I, \dot{\theta}^I, \Psi) - \sum_I \dot{\theta}^I p_{\theta^I}. \quad (3.24)$$

Due to the partial Legendre transformation, the variables r^I and Ψ still follow the Euler–Lagrange equations while the ignorable coordinates θ^I and their momenta $h^I := p_{\theta^I}$ obey the Hamilton equations. However, the latter set of equations results in trivial statements; the constant property of h^I (i.e. $\dot{h}^I = 0$) and the definition of h^I (i.e. $\dot{\theta}^I = \frac{h^I}{(r^I)^2}$). So classically the Routhian is not really $R(r^I, \dot{r}^I, h^I, \Psi)$ but rather $R(r^I, \dot{r}^I, \Psi)$ along with the integrals of motion h^I 's. Hence we can rewrite (3.23) as

$$S = \int_{\mathbb{R}} dt \left[\frac{1}{2} \sum_I (\dot{r}^I)^2 - \frac{1}{2} \sum_I \frac{(h^I)^2}{(r^I)^2} - \frac{i}{2} \bar{\Psi}^{\alpha a} \dot{\Psi}_{\alpha a} + \lambda \left(kC_1(E) + \sum_I h^I + i\bar{\Psi}_A^\alpha \Psi_{\alpha B} \right) \right]. \quad (3.25)$$

³ The same result can be obtained from the Hamiltonian reduction by rewriting the Hamiltonian in terms of the conserved charges.

Integrating out λ , we finally obtain the reduced effective action

$$S = \frac{1}{2} \int_{\mathbb{R}} dt \left[\dot{q}^2 + \sum_{I \neq K} (\dot{r}^I)^2 - i \bar{\Psi}^{\alpha a} \dot{\Psi}_{\alpha a} - \frac{[kC_1(E) + \sum_{I \neq K} h^I + i \bar{\Psi}_A^\alpha \Psi_{\alpha B}]^2}{q^2} - \sum_{I \neq K} \frac{(h^I)^2}{(r^I)^2} \right]. \quad (3.26)$$

Here we have defined $q := r^K$ where K denotes the specific direction in which h^K is automatically determined by other conserved quantities h^I 's. Note that the terms appearing in the numerator of the potential are the integrals of motion, namely they commute with the Hamiltonian.

Let us study the classical properties of the theory (3.26). The action (3.26) leads to the classical equations of motion

$$\ddot{q} = \frac{[kC_1(E) + \sum_{I \neq K} h^I + i \bar{\Psi}_A^\alpha \Psi_{\alpha B}]^2}{q^3} \quad (3.27)$$

$$\ddot{r}^I = \frac{(h^I)^2}{(r^I)^3} \quad (3.28)$$

$$\dot{\Psi}_{\alpha A} = - \frac{[kC_1(E) + \sum_{I \neq K} h^I + i \bar{\Psi}_A^\alpha \Psi_{\alpha B}]}{q^2} \Psi_{\alpha B} \quad (3.29)$$

$$\dot{\Psi}_{\alpha B} = \frac{[kC_1(E) + \sum_{I \neq K} h^I + i \bar{\Psi}_A^\alpha \Psi_{\alpha B}]}{q^2} \Psi_{\alpha A}. \quad (3.30)$$

Making use of the equations of motion (3.29) and (3.30), one can check that the differentiation of the Gauss law constraint (3.22) with respect to time t vanishes. In other words, ϕ_0 is the constant of motion.

The canonical momenta are

$$p := \frac{\partial L}{\partial \dot{q}} = \dot{q}, \quad p_I := \frac{\partial L}{\partial \dot{r}^I} = \dot{r}^I \quad (3.31)$$

$$\pi^{\alpha a} := \frac{\vec{\partial} L}{\partial \dot{\Psi}_{\alpha a}} = \frac{i}{2} \bar{\Psi}^{\alpha a}, \quad \bar{\pi}_{\alpha a} := \frac{\vec{\partial} L}{\partial \dot{\bar{\Psi}}^{\alpha a}} = \frac{i}{2} \Psi_{\alpha a}. \quad (3.32)$$

The fermionic momenta $\pi^{\alpha a}$ and $\bar{\pi}_{\alpha a}$ do not depend on the velocities but on the fermionic degrees of freedom themselves. Hence one can read second-class constraints

$$\phi_1^{\alpha a} := \pi^{\alpha a} - \frac{i}{2} \bar{\Psi}^{\alpha a} = 0, \quad \phi_{2\alpha a} := \bar{\pi}_{\alpha a} - \frac{i}{2} \Psi_{\alpha a} = 0. \quad (3.33)$$

Under the constraints, we get the Dirac brackets

$$[q, p]_{DB} = 1, \quad [r^I, p_J]_{DB} = \delta^I_J \quad (3.34)$$

$$[\Psi_{\alpha A}, \pi^{\beta b \dot{B}}]_{DB} = -\frac{1}{2} \delta_{\alpha\beta} \delta_{ab} \delta_{\dot{A}\dot{B}}, \quad [\Psi_{\alpha A}, \bar{\Psi}^{\beta b \dot{B}}]_{DB} = i \delta_{\alpha\beta} \delta_{ab} \delta_{\dot{A}\dot{B}}. \quad (3.35)$$

The action (3.26) is invariant under the following one-dimensional conformal transformations

$$\delta t = f(t) = a + bt + ct^2, \quad \delta \partial_0 = -\dot{f} \partial_0 \quad (3.36)$$

$$\delta q = \frac{1}{2} \dot{f} q, \quad \delta r^I = \frac{1}{2} \dot{f} r^I \quad (3.37)$$

$$\delta \Psi_{\alpha a} = 0, \quad \delta \bar{\Psi}^{\alpha a} = 0. \quad (3.38)$$

Here the constant parameters a , b and c are infinitesimal parameters of translation, dilatation and conformal boost respectively. The corresponding Noether charges, the Hamiltonian H , the dilatation operator D and the conformal boost operator K are found to be

$$H = \frac{1}{2} \left[p^2 + \frac{(kC_1(E) + \sum_{I \neq K} h^I + i\bar{\Psi}_A^\alpha \Psi_{\alpha B})^2}{q^2} + \sum_{I \neq K} \left(p_I^2 + \frac{(h^I)^2}{(r^I)^2} \right) \right] \quad (3.39)$$

$$D = tH - \frac{1}{4} \left[(qp + pq) + \sum_{I \neq K} (r^I p_I + p_I r^I) \right] \quad (3.40)$$

$$K = t^2 H - \frac{1}{2} t \left[(qp + pq) + \sum_{I \neq K} (r^I p_I + p_I r^I) \right] + \frac{1}{2} \left[q^2 + \sum_{I \neq K} (r^I)^2 \right]. \quad (3.41)$$

The operators D and K are the constants of motion in the sense that

$$\frac{\partial D}{\partial t} + [H, D]_{DB} = 0, \quad \frac{\partial K}{\partial t} + [H, K]_{DB} = 0. \quad (3.42)$$

Note that the explicit time dependence of D and K can be absorbed into the similarity transformations

$$D = e^{itH} D_0 e^{-itH}, \quad K = e^{itH} K_0 e^{-itH} \quad (3.43)$$

where

$$D_0 := -\frac{1}{4} \left[(qp + pq) + \sum_{I \neq K} (r^I p_I + p_I r^I) \right] \quad (3.44)$$

$$K_0 := \frac{1}{2} \left[q^2 + \sum_{I \neq K} (r^I)^2 \right] \quad (3.45)$$

are time independent parts of D and K respectively. So we will use the time independent parts as the explicit expressions for D and K and drop off the subscripts.

The action (3.26) is invariant under the following fermionic transformations

$$\delta q = \frac{i}{\sqrt{2}} (\bar{\epsilon}^- \Psi_{-A} - \bar{\epsilon}^+ \Psi_{+A}) + \frac{i}{\sqrt{2}} (\bar{\epsilon}^- \Psi_{-B} - \bar{\epsilon}^+ \Psi_{+B}) \quad (3.46)$$

$$\delta r^I = i \cos \theta^I (\bar{\epsilon}^+ \tilde{r}^I \Psi_{-A} - \bar{\epsilon}^- \tilde{r}^I \Psi_{+A}) + i \sin \theta^I (\bar{\epsilon}^+ \tilde{r}^I \Psi_{-B} - \bar{\epsilon}^- \tilde{r}^I \Psi_{+B}) \quad (3.47)$$

$$\delta \Psi_{+A\dot{A}} = -\frac{1}{\sqrt{2}} \left(\dot{q} - \frac{h^K}{q} \right) \epsilon_{+\dot{A}} - \frac{i}{\sqrt{2}} \frac{l}{q} \Psi_{+B\dot{A}} - \sum_{I \neq K} \left(\dot{r}^I \cos \theta^I - \sin \theta^I \frac{h^I}{r^I} \right) \tilde{r}^I \epsilon_{-\dot{A}} \quad (3.48)$$

$$\delta \Psi_{-A\dot{A}} = \frac{1}{\sqrt{2}} \left(\dot{q} - \frac{h^K}{q} \right) \bar{\epsilon}_{-\dot{A}} - \frac{i}{\sqrt{2}} \frac{l}{q} \Psi_{-B\dot{A}} + \sum_{I \neq K} \left(\dot{r}^I \cos \theta^I - \sin \theta^I \frac{h^I}{r^I} \right) \tilde{r}^I \epsilon_{+\dot{A}} \quad (3.49)$$

$$\delta\psi_{+B\dot{A}} = -\frac{1}{\sqrt{2}}\left(\dot{q} + \frac{h^K}{q}\right)\epsilon_{+\dot{A}} + \frac{i}{\sqrt{2}}\frac{l}{q}\psi_{+A\dot{A}} - \sum_{I \neq K} \left(\dot{r}^I \sin\theta^I + \cos\theta^I \frac{h^I}{r^I}\right) \tilde{r}^I \epsilon_{-\dot{A}} \quad (3.50)$$

$$\delta\psi_{-B\dot{A}} = \frac{1}{\sqrt{2}}\left(\dot{q} + \frac{h^K}{q}\right)\tilde{\epsilon}_{-\dot{A}} + \frac{i}{\sqrt{2}}\frac{l}{q}\psi_{-A\dot{A}} + \sum_{I \neq K} \left(\dot{r}^I \sin\theta^I + \cos\theta^I \frac{h^I}{r^I}\right) \tilde{r}^I \epsilon_{+\dot{A}} \quad (3.51)$$

where we have defined

$$\theta^I(t) = h^I \int^t \frac{dt'}{(r^I(t'))^2} \quad (3.52)$$

$$l = (\bar{\Psi}^{+A}\epsilon_+ - \bar{\Psi}^{-A}\epsilon_-) - (\bar{\Psi}^{+B}\epsilon_+ - \bar{\Psi}^{-B}\epsilon_-). \quad (3.53)$$

We should note that the supersymmetry is generically non-local in the sense that the transformations contain the integrals of the function of the non-local variables r^I 's with respect to time. The non-locality is the consequence of the Routh reduction. Hence the infinite number of the associated conserved charges may exist and things may become much more exotic. However, as seen from the (3.39), the motion in the K -th direction endowed with the local supersymmetry and others with non-local ones are essentially decoupled because their Hamiltonians commute with each other. Thus we can treat them separately. This indicates that the theory possesses the local conserved supercurrents and the non-local supercurrents which are in involution.

3.1.3. $OSp(16|2)$ superconformal mechanics

Now we want to study the K -th motion associated with the local charges and shed light on the algebraic structure of the symmetry group in the quantum mechanics. For simplicity let us consider the case where the all independent conserved charges h^I 's are zeros. This is realized when the internal degrees of freedom for two M2-branes are unbiased.⁴

The dynamics in the K -th direction is given by the action

$$S = \frac{1}{2} \int_{\mathbb{R}} dt \left[\dot{q}^2 - i\bar{\Psi}^{\alpha a} \dot{\Psi}_{\alpha a} - \frac{(kC_1(E) + i\bar{\Psi}_A^\alpha \Psi_{\alpha B})^2}{q^2} \right]. \quad (3.54)$$

The notable feature is that our reduced action (3.54) has the predicted form for $\mathcal{N} > 4$ superconformal mechanical action discussed in [16–18].

The action (3.54) is invariant under the following $\mathcal{N} = 16$ supersymmetry transformation laws

$$\delta q = \frac{i}{\sqrt{2}}(\bar{\epsilon}^- \Psi_{-A} - \bar{\epsilon}^+ \Psi_{+A}) + \frac{i}{\sqrt{2}}(\bar{\epsilon}^- \Psi_{-B} - \bar{\epsilon}^+ \Psi_{+B}) \quad (3.55)$$

$$\delta\psi_{+A\dot{A}} = -\frac{1}{\sqrt{2}}\left(\dot{q} + \frac{g}{q}\right)\epsilon_{+\dot{A}} - \frac{i}{\sqrt{2}}\frac{l}{q}\psi_{+B\dot{A}} \quad (3.56)$$

$$\delta\psi_{-A\dot{A}} = \frac{1}{\sqrt{2}}\left(\dot{q} + \frac{g}{q}\right)\epsilon_{-\dot{A}} - \frac{i}{\sqrt{2}}\frac{l}{q}\psi_{-B\dot{A}} \quad (3.57)$$

⁴ This specific charge assignment does not affect the following discussion for the K -th motion since non-vanishing h^I 's can only give rise to a constant shift in the numerator of the potential.

$$\delta\Psi_{+B\dot{A}} = -\frac{1}{\sqrt{2}}\left(\dot{q} - \frac{g}{q}\right)\epsilon_{+\dot{A}} + \frac{i}{\sqrt{2}}\frac{l}{q}\Psi_{+A\dot{A}} \quad (3.58)$$

$$\delta\Psi_{-B\dot{A}} = \frac{1}{\sqrt{2}}\left(\dot{q} - \frac{g}{q}\right)\epsilon_{-\dot{A}} + \frac{i}{\sqrt{2}}\frac{l}{g}\Psi_{-A\dot{A}} \quad (3.59)$$

where

$$g = kC_1(E) + i\bar{\Psi}_A^\alpha\Psi_{\alpha B}. \quad (3.60)$$

These supersymmetry transformations are local and we therefore can apply the conventional Noether's procedure. By means of the Noether's method, the corresponding supercharges are calculated to be

$$Q_{+\dot{A}} = \frac{i}{\sqrt{2}}\left(p + \frac{g}{q}\right)\Psi_{+A\dot{A}} + \frac{i}{\sqrt{2}}\left(p - \frac{g}{q}\right)\Psi_{+B\dot{A}} \quad (3.61)$$

$$Q_{-\dot{A}} = -\frac{i}{\sqrt{2}}\left(p + \frac{g}{q}\right)\Psi_{-A\dot{A}} - \frac{i}{\sqrt{2}}\left(p - \frac{g}{q}\right)\Psi_{-B\dot{A}}. \quad (3.62)$$

Since the action (3.54) is invariant under the conformal transformations $\delta t = f(t)$, $\delta q = \frac{1}{2}\dot{f}q$ and $\delta\Psi_{\alpha a} = 0$, three generators, the Hamiltonian H , the dilatation generator D and the conformal boost generator K are explicitly expressed as

$$H = \frac{1}{2}p^2 + \frac{[kC_1(E) + i\bar{\Psi}_A^\alpha\Psi_{\alpha B}]^2}{2q^2} \quad (3.63)$$

$$D = -\frac{1}{4}\{q, p\} \quad (3.64)$$

$$K = \frac{1}{2}q^2 \quad (3.65)$$

where $\{, \}$ represents an anti-commutator.

In order to quantize the theory, we impose the (anti)commutation relations for the canonical variables obtained from the Dirac brackets (3.34) and (3.35)

$$[q, p] = i, \quad \{\Psi_{\alpha a\dot{A}}, \bar{\Psi}^{\beta b\dot{B}}\} = -\delta_{\alpha\beta}\delta_{ab}\delta_{\dot{A}\dot{B}}. \quad (3.66)$$

The presence of the conformal symmetry and the supersymmetry leads to that of a superconformal symmetry. Let us define the superconformal boost generators

$$S_{+\dot{A}} = \frac{i}{\sqrt{2}}q(\Psi_{+A\dot{A}} + \Psi_{+B\dot{A}}) \quad (3.67)$$

$$S_{-\dot{A}} = -\frac{i}{\sqrt{2}}q(\Psi_{-A\dot{A}} + \Psi_{-B\dot{A}}). \quad (3.68)$$

Additionally the theory has the internal R-symmetry which rotates the fermionic charges. We define the R-symmetry generators by

$$(J_{\alpha\beta})_{\dot{A}\dot{B}} = (J_{\alpha\beta})_{\dot{A}\dot{B}}^{(1)} + (J_{\alpha\beta})_{\dot{A}\dot{B}}^{(2)} \quad (3.69)$$

where

$$(J_{\alpha\beta})_{\dot{A}\dot{B}}^{(1)} = i\Psi_{\alpha a\dot{A}}\bar{\Psi}^{\beta a\dot{B}}, \quad (J_{\alpha\beta})_{\dot{A}\dot{B}}^{(2)} = i\bar{\Psi}^{\alpha a\dot{A}}\Psi_{\beta a\dot{B}}. \quad (3.70)$$

Notice that the R-symmetry generators satisfy the relations

$$(J_{++})_{\dot{A}\dot{B}} = -(J_{++})_{\dot{B}\dot{A}} \quad (3.71)$$

$$(J_{--})_{\dot{A}\dot{B}} = -(J_{--})_{\dot{B}\dot{A}} \quad (3.72)$$

$$(J_{+-})_{\dot{A}\dot{B}} = -(J_{+-})_{\dot{B}\dot{A}} \quad (3.73)$$

and therefore the matrices J_{++} , J_{--} and J_{+-} contain 28, 28 and 64 independent entries respectively while J_{-+} yields no independent ones because of the relations (3.73). Therefore the R-symmetry matrix totally carries $28 + 28 + 64 = 120$ elements.

Using the canonical (anti)commutation relations (3.66), one can find the complete set of (anti)commutators among the generators

$$[H, D] = iH, \quad [K, D] = -iK, \quad [H, K] = 2iD \quad (3.74)$$

$$[(J_{\alpha\beta})_{\dot{A}\dot{B}}, H] = 0, \quad [(J_{\alpha\beta})_{\dot{A}\dot{B}}, D] = 0, \quad [(J_{\alpha\beta})_{\dot{A}\dot{B}}, K] = 0 \quad (3.75)$$

$$[(J_{\alpha\beta})_{\dot{A}\dot{B}}, (J_{\gamma\delta})_{\dot{C}\dot{D}}] = i(J_{\gamma\beta})_{\dot{C}\dot{B}}\delta_{\alpha\delta}\delta_{\dot{A}\dot{D}} - i(J_{\alpha\delta})_{\dot{A}\dot{D}}\delta_{\beta\gamma}\delta_{\dot{B}\dot{C}} \\ + i(J_{\delta\beta})_{\dot{D}\dot{B}}\delta_{\alpha\gamma}\delta_{\dot{A}\dot{C}} - i(J_{\alpha\gamma})_{\dot{A}\dot{C}}\delta_{\beta\delta}\delta_{\dot{B}\dot{D}} \quad (3.76)$$

$$[H, Q_{\alpha\dot{A}}] = 0, \quad [D, Q_{\alpha\dot{A}}] = -\frac{i}{2}Q_{\alpha\dot{A}}, \quad [K, Q_{\alpha\dot{A}}] = iS_{\alpha\dot{A}} \quad (3.77)$$

$$[H, \bar{Q}^{\alpha\dot{A}}] = 0, \quad [D, \bar{Q}^{\alpha\dot{A}}] = -\frac{i}{2}\bar{Q}^{\alpha\dot{A}}, \quad [K, \bar{Q}^{\alpha\dot{A}}] = i\bar{S}^{\alpha\dot{A}} \quad (3.77)$$

$$[H, S_{\alpha\dot{A}}] = -iQ_{\alpha\dot{A}}, \quad [D, S_{\alpha\dot{A}}] = \frac{i}{2}S_{\alpha\dot{A}}, \quad [K, S_{\alpha\dot{A}}] = 0 \quad (3.78)$$

$$[H, \bar{S}^{\alpha\dot{A}}] = -i\bar{Q}^{\alpha\dot{A}}, \quad [D, \bar{S}^{\alpha\dot{A}}] = \frac{i}{2}\bar{S}^{\alpha\dot{A}}, \quad [K, \bar{S}^{\alpha\dot{A}}] = 0 \quad (3.78)$$

$$\{Q_{\alpha\dot{A}}, \bar{Q}^{\beta\dot{B}}\} = 2H\delta_{\alpha\beta}\delta_{\dot{A}\dot{B}} \\ \{S_{\alpha\dot{A}}, \bar{S}^{\beta\dot{B}}\} = 2K\delta_{\alpha\beta}\delta_{\dot{A}\dot{B}} \\ \{Q_{\alpha\dot{A}}, \bar{S}^{\beta\dot{B}}\} = -2D\delta_{\alpha\beta}\delta_{\dot{A}\dot{B}} + (J_{\alpha\beta})_{\dot{A}\dot{B}}^{(1)}(\delta_{\alpha\beta} - \delta_{\alpha-\beta}) - \frac{i}{2}\delta_{\alpha\beta}\delta_{\dot{A}\dot{B}} \\ \{\bar{Q}^{\alpha\dot{A}}, S_{\beta\dot{B}}\} = -2D\delta_{\alpha\beta}\delta_{\dot{A}\dot{B}} + (J_{\alpha\beta})_{\dot{A}\dot{B}}^{(2)}(\delta_{\alpha\beta} - \delta_{\alpha-\beta}) - \frac{i}{2}\delta_{\alpha\beta}\delta_{\dot{A}\dot{B}} \quad (3.79)$$

$$[(J_{\alpha\beta})_{\dot{A}\dot{B}}, Q_{\gamma\dot{C}}] = i(Q_{\beta\dot{B}}\delta_{\alpha\gamma}\delta_{\dot{A}\dot{C}} - Q_{\alpha\dot{A}}\delta_{\beta\gamma}\delta_{\dot{B}\dot{C}}) \\ [(J_{\alpha\beta})_{\dot{A}\dot{B}}, S_{\gamma\dot{C}}] = i(S_{\beta\dot{B}}\delta_{\alpha\gamma}\delta_{\dot{A}\dot{C}} - S_{\alpha\dot{A}}\delta_{\beta\gamma}\delta_{\dot{B}\dot{C}}) \\ [(J_{\alpha\beta})_{\dot{A}\dot{B}}, \bar{Q}^{\gamma\dot{C}}] = -i(\bar{Q}^{\beta\dot{B}}\delta_{\alpha\gamma}\delta_{\dot{A}\dot{C}} - \bar{Q}^{\alpha\dot{A}}\delta_{\beta\gamma}\delta_{\dot{B}\dot{C}}) \\ [(J_{\alpha\beta})_{\dot{A}\dot{B}}, \bar{S}^{\gamma\dot{C}}] = -i(\bar{S}^{\beta\dot{B}}\delta_{\alpha\gamma}\delta_{\dot{A}\dot{C}} - \bar{S}^{\alpha\dot{A}}\delta_{\beta\gamma}\delta_{\dot{B}\dot{C}}). \quad (3.80)$$

The Hamiltonian H , the dilatation generator D and the conformal boost generator K satisfy the one-dimensional conformal algebra (3.74). By defining

$$T_0 = \frac{1}{2}\left(\frac{K}{a} + aH\right) \quad (3.81)$$

$$T_1 = D \quad (3.82)$$

$$T_2 = \frac{1}{2}\left(\frac{K}{a} - aH\right) \quad (3.83)$$

with a being a constant with dimension of length, one finds the explicit representation of the $\mathfrak{so}(1, 2)$ algebra

$$[T_i, T_j] = i\epsilon_{ijk} T^k \quad (3.84)$$

where ϵ_{ijk} is a three-index anti-symmetric tensor with $\epsilon_{012} = 1$ and $g_{ij} = \text{diag}(1, -1, -1)$. Alternatively, if we introduce

$$L_0 = \frac{1}{2} \left(aH + \frac{K}{a} \right) \quad (3.85)$$

$$L_{\pm} = \frac{1}{2} \left(aH - \frac{K}{a} \pm 2iD \right), \quad (3.86)$$

then we get the explicit representation of the $\mathfrak{sl}(2, \mathbb{R})$ algebra in the Virasoro form

$$[L_m, L_n] = (m - n)L_{m+n} \quad (3.87)$$

with $m, n = 0, \pm 1$.

As the superpartners of the conformal generators there are sixteen supercharges $Q_{\alpha\dot{A}}$ and as many superconformal generators $S_{\alpha\dot{A}}$. As seen from (3.75) and (3.80), the R-symmetry generators $(J_{\alpha\beta})_{\dot{A}\dot{B}}$ commute with the bosonic generators H , D and K while they yield the rotations of the fermionic generators $Q_{\alpha\dot{A}}$ and $S_{\alpha\dot{A}}$. The commutation relation (3.76) implies that $(J_{\alpha\beta})_{\dot{A}\dot{B}}$ obey the $\mathfrak{so}(16)$ algebra. Therefore we can conclude that the theory (3.54) is the $OSp(16|2)$ invariant $\mathcal{N} = 16$ superconformal mechanics. Indeed this fits in the list of the possible simple supergroup for superconformal quantum mechanics [33,34].

We see that the R-symmetry is now enhanced in our quantum mechanics. Interestingly a similar phenomenon has been already observed in $d = 11$ supergravity. In $d = 11$ supergravity the original tangent space symmetry $SO(1, 10)$ can break down into the subgroup $SO(1, 2) \times SO(8)$ through a partial choice of gauge for the elfbein. However, it has been pointed out in [35–37] that one can find the enhanced $SO(1, 2) \times SO(16)$ tangent space symmetry by introducing new gauge degrees of freedom. It would be intriguing to inquire whether the enlarged R-symmetry of our quantum mechanics reflects that of $d = 11$ supergravity.

3.2. $\mathcal{N} = 12$ superconformal mechanics

3.2.1. Derivation of quantum mechanics

Let us consider the $U(N)_k \times \hat{U}(N)_{-k}$ ABJM-model on $\mathbb{R} \times T^2$. The theory may describe the dynamics of N coincident M2-branes with the world-volume $M_3 = \mathbb{R} \times T^2$ moving in a transverse space with an $SU(4)$ holonomy. We now want to derive the low-energy effective theory describing the dynamics around static BPS configurations. Such BPS configurations obey the BPS equations. From the supersymmetry transformations (2.32), (2.33) for fermions we find the following set of BPS equations:

$$D_z Y^A = 0, \quad D_{\bar{z}} Y^A = 0 \quad (3.88)$$

$$Y^C Y_C^\dagger Y^B - Y^B Y_C^\dagger Y^C = 0 \quad (3.89)$$

$$Y^C Y_A^\dagger Y^D = 0. \quad (3.90)$$

To satisfy the algebraic equations (3.89) and (3.90), the bosonic Higgs fields Y^A and Y_A^\dagger should take the diagonal form

$$Y^A = \text{diag}(y_1^A, \dots, y_N^A), \quad Y_A^\dagger = \text{diag}(\bar{y}_{A1}, \dots, \bar{y}_{AN}) \quad (3.91)$$

where y_a^A is a complex scalar field. For the above diagonal configurations, all the off-diagonal elements are massive and the gauge group $U(N) \times \hat{U}(N)$ is spontaneously broken to $U(1)^N$ [9]. Let us define

$$\mathcal{A}_{\mu a}^+ := A_{\mu aa} + \hat{A}_{\mu aa}, \quad \mathcal{A}_{\mu a}^- := A_{\mu aa} - \hat{A}_{\mu aa} \quad (3.92)$$

where the indices $a = 1, \dots, N$ characterize the gauge degrees of freedom, i.e. the internal degrees of freedom of the multiple M2-branes. Note that all the couplings involve the gauge fields $\mathcal{A}_{\mu a}^-$ while the other gauge fields $\mathcal{A}_{\mu a}^+$ are associated with the preserved $U(1)$ gauge group. In terms of the expressions (3.91) and (3.92), we can rewrite Eqs. (3.88) as

$$\partial_z y_a^A + i \mathcal{A}_{za}^- y_a^A = 0, \quad \partial_z \bar{y}_{Aa} - i \mathcal{A}_{za}^- \bar{y}_{Aa} = 0 \quad (3.93)$$

$$\partial_{\bar{z}} y_a^A + i \mathcal{A}_{\bar{z}a}^- y_a^A = 0, \quad \partial_{\bar{z}} \bar{y}_{Aa} - i \mathcal{A}_{\bar{z}a}^- \bar{y}_{Aa} = 0 \quad (3.94)$$

$$A_{zab} = \hat{A}_{zab} = A_{\bar{z}ab} = \hat{A}_{\bar{z}ab} = 0 \quad \text{for } a \neq b. \quad (3.95)$$

The first and second lines correspond to the equations for diagonal elements and last one is for the off-diagonal elements. The general solutions to Eqs. (3.93) and (3.94) are given by

$$y_a^A = r_a^A e^{i(\varphi_a(z, \bar{z}) + \theta_a^A)} \quad (3.96)$$

$$\mathcal{A}_{za}^- = -\partial_z \varphi_a(z, \bar{z}) \quad (3.97)$$

where $r_a^A, \theta_a^A \in \mathbb{R}$ have no dependence on z and \bar{z} while $\varphi_a(z, \bar{z}) \in \mathbb{R}$ is a function of z and \bar{z} . The expression (3.97) ensures the flatness of the $U(1)$ gauge field \mathcal{A}_z^- . Hence $\varphi_a, \mathcal{A}_{za}^-$ and $\mathcal{A}_{\bar{z}a}^-$ take the form [26]

$$\varphi_a(z, \bar{z}) = -2\pi \frac{\Theta_a}{\tau - \bar{\tau}} z + 2\pi \frac{\bar{\Theta}_a}{\tau - \bar{\tau}} \bar{z} \quad (3.98)$$

$$\mathcal{A}_{za}^- = 2\pi \frac{\Theta_a}{\tau - \bar{\tau}} \omega, \quad \mathcal{A}_{\bar{z}a}^- = -2\pi \frac{\bar{\Theta}_a}{\tau - \bar{\tau}} \bar{\omega}. \quad (3.99)$$

Here τ is the moduli of the torus defined in (3.1) and $\Theta_a := \zeta_a + \bar{\tau} \xi_a$, $a = 1, \dots, N$, are the coordinates of the product space of the N Jacobi varieties characterizing the N $U(1)$ flat bundles. For the bosonic Higgs fields to describe the positions of the membranes, we should impose the single-valuedness of y_a^A as

$$\begin{aligned} y_a^A(z+1, \bar{z}+1) &= y_a^A(z, \bar{z}) \\ y_a^A(z+\tau, \bar{z}+\bar{\tau}) &= y_a^A(z, \bar{z}). \end{aligned} \quad (3.100)$$

These conditions require that ξ_a and ζ_a are integers, which result in the quantization of the variables $\varphi_a, \mathcal{A}_{za}^-$ and $\mathcal{A}_{\bar{z}a}^-$. Then the resulting static BPS configurations are

$$\begin{aligned} Y^A &= \text{diag}(y_1^A, \dots, y_N^A) = \text{diag}(r_1^A e^{i(\varphi_1(z, \bar{z}) + \theta_1^A)}, \dots, r_N^A e^{i(\varphi_N(z, \bar{z}) + \theta_N^A)}) \\ Y_A^\dagger &= \text{diag}(\bar{y}_{A1}, \dots, \bar{y}_{AN}) = \text{diag}(r_1^A e^{-i(\varphi_1(z, \bar{z}) + \theta_1^A)}, \dots, r_N^A e^{-i(\varphi_N(z, \bar{z}) + \theta_N^A)}) \\ A_z &= \text{diag}(A_{z11}, \dots, A_{zNN}) \\ \hat{A}_z &= A_z + \partial_z \varphi = \text{diag}(A_{z11} + \partial_z \varphi_1, \dots, A_{zNN} + \partial_z \varphi_N). \end{aligned} \quad (3.101)$$

By the supersymmetry the above bosonic configurations are paired with the fermionic fields

$$\psi_{\pm A} = \text{diag}(\psi_{\pm A1}, \dots, \psi_{\pm AN}), \quad \psi_{\pm}^{\dagger A} = \text{diag}(\psi_{\pm}^{\dagger A1}, \dots, \psi_{\pm}^{\dagger AN}) \quad (3.102)$$

where the subscripts \pm label the $SO(2)_E$ spinor representation.

Inserting the set of BPS configurations (3.101) and (3.102) into the ABJM action (2.26) one finds

$$S = \int_{\mathbb{R}} dt \int_{T^2} d^2z \sum_A \sum_{a=1}^N \left[D_0 \bar{y}_A^a D_0 y_a^A - i \psi_+^{\dagger Aa} D_0 \psi_{+Aa} - i \psi_-^{\dagger Aa} D_0 \psi_{-Aa} \right. \\ \left. + \frac{k}{4\pi} \left(\mathcal{A}_{0a}^- \mathcal{F}_{z\bar{z}a}^+ + \frac{1}{2} \mathcal{A}_{\bar{z}a}^- \dot{\mathcal{A}}_{za}^+ - \frac{1}{2} \mathcal{A}_{za}^- \dot{\mathcal{A}}_{\bar{z}a}^+ \right) \right]. \quad (3.103)$$

Recall that $\mathcal{A}_{\bar{z}}^-$ and $\mathcal{A}_{\bar{z}}^+$ are quantized and their time derivative terms do not show up in the action. Thus we can treat them as auxiliary fields and integrate out them. Consequently we get constraints $\dot{\mathcal{A}}_{za}^+ = \dot{\mathcal{A}}_{\bar{z}a}^+ = 0$, which imply that the gauge fields \mathcal{A}_{za}^+ and $\mathcal{A}_{\bar{z}a}^+$ on the Riemann surface have no time dependence.

Taking these constraints into account and proceeding the integration over the torus, we obtain the low-energy effective action

$$S = \int_{\mathbb{R}} dt \left[D_0 \bar{y}_A^a D_0 y_a^A - i \psi^{\dagger \alpha Aa} D_0 \psi_{\alpha Aa} + k C_1(E_a) \mathcal{A}_{0a}^- \right]. \quad (3.104)$$

Here the repeated indices are summed over and $\alpha, \beta, \dots = +, -$ denote the $SO(2)_E$ spinor indices. The covariant derivatives are defined by

$$D_0 y_a^A = \dot{y}_a^A + i \mathcal{A}_{0a}^- y_a^A, \quad D_0 \bar{y}_{Aa} = \dot{\bar{y}}_{Aa} - i \mathcal{A}_{0a}^- \bar{y}_{Aa} \\ D_0 \psi_{\alpha Aa} = \dot{\psi}_{\alpha Aa} + i \mathcal{A}_{0a}^- \psi_{\alpha Aa}, \quad D_0 \psi_{\alpha a}^{\dagger A} = \dot{\psi}_{\alpha a}^{\dagger A} - i \mathcal{A}_{0a}^- \psi_{\alpha a}^{\dagger A} \quad (3.105)$$

and

$$C_1(E_a) := \frac{1}{2\pi} \int_{T^2} F_{z\bar{z}aa} = \frac{1}{4\pi} \int_{T^2} \mathcal{F}_{z\bar{z}a}^+ \quad (3.106)$$

is the Chern number of the a -th $U(1)$ principal bundle $E_a \rightarrow T^2$ over the torus associated with the preserved $U(1)$ gauge fields $\mathcal{A}_{z\bar{z}aa}$.

The action (3.104) is invariant under the one-dimensional conformal transformations

$$\delta t = f(t) = a + bt + ct^2, \quad \delta \partial_0 = -\dot{f} \partial_0 \quad (3.107)$$

$$\delta y_a^A = \frac{1}{2} \dot{f} y_a^A, \quad \delta \bar{y}_{Aa} = \frac{1}{2} \dot{f} \bar{y}_{Aa} \quad (3.108)$$

$$\delta \psi_{\alpha Aa} = 0, \quad \delta \psi_{\alpha a}^{\dagger A} = 0 \quad (3.109)$$

$$\delta \mathcal{A}_{0a}^- = -\dot{f} \mathcal{A}_{0a}^- \quad (3.110)$$

and $\mathcal{N} = 12$ supersymmetry transformations

$$\delta y_a^A = i \omega^{\alpha AB} \psi_{\alpha Ba}, \quad \delta \bar{y}_{Aa} = i \psi_a^{\dagger \alpha B} \omega_{\alpha AB} \quad (3.111)$$

$$\delta \psi_{\alpha Aa} = \omega_{\alpha AB} D_0 y_a^B, \quad \delta \psi_{\alpha a}^{\dagger A} = -D_0 \bar{y}_{Ba} \omega_{\alpha}^{AB} \quad (3.112)$$

$$\delta \mathcal{A}_{0a}^- = 0 \quad (3.113)$$

where the supersymmetry parameters $\omega_{+AB} := \epsilon_{+i} (\Gamma^i)_{AB}$ and $\omega_{-AB} := \epsilon_{-i} (\Gamma^i)_{AB}$ transform as $\mathbf{6}_+$ and $\mathbf{6}_-$ under $SU(4) \times SO(2)_E$ respectively. Therefore the low-energy effective theory is described by the $\mathcal{N} = 12$ superconformal gauged quantum mechanics (3.104).

3.2.2. Reduced system with inverse-square interaction

The low-energy effective action (3.104) is quadratic in \mathcal{A}_{0a}^- and contains no time derivatives of \mathcal{A}_{0a}^- . So they are auxiliary fields and we want to integrate them out. Let us fix the gauge as $\mathcal{A}_{0a}^- = 0$. Then the algebraic equations of motion of \mathcal{A}_{0a}^- yield the Gauss law constraints, the moment map conditions

$$\phi_{0a} := kC_1(E_a) + 2 \sum_A (r_a^A)^2 \dot{\theta}_a^A + \sum_A \psi^{\dagger \alpha A a} \psi_{\alpha A a} = 0 \quad (3.114)$$

for $a = 1, \dots, N$. Note that although the set of equations (3.114) has the same form as that of (3.22), the physical meaning of these constraints are different because the angular variable θ_a^A 's are defined not in the abstract space of the internal degrees of freedom as in (3.22), but in the actual configuration space of the a -th M2-brane in the A -th complex plane.

Defining the conserved charges $h_a^A := 2(r_a^A)^2 \dot{\theta}_a^A$, using the above constraints (3.114) and following the reduction procedure as in the derivation of (3.26), we can integrate out the auxiliary gauge fields \mathcal{A}_{0a}^- and find the reduced effective action with the inverse-square type interaction

$$\begin{aligned} S = \int_{\mathbb{R}} dt \sum_{a=1}^N & \left[\dot{x}_a^2 - \frac{i}{2} \sum_{A \neq B} (\psi^{\dagger \alpha A a} \dot{\psi}_{\alpha A a} - \dot{\psi}^{\dagger A a} \psi_{\alpha A a}) \right. \\ & + \sum_{A \neq B} (\dot{r}_a^A)^2 - \frac{i}{2} (\lambda^{\dagger \alpha a} \dot{\lambda}_{\alpha a} - \dot{\lambda}^{\dagger \alpha a} \lambda_{\alpha a}) \\ & \left. - \frac{[kC_1(E_a) + \sum_{A \neq B} h_a^A + \sum_{A \neq B} \psi^{\dagger \alpha A a} \psi_{\alpha A a} + \lambda^{\dagger \alpha a} \lambda_{\alpha a}]^2}{4x_a^2} - \sum_{A \neq B} \frac{(h_a^A)^2}{4(r_a^A)^2} \right]. \end{aligned} \quad (3.115)$$

Here $x_a := r_a^B$ describes the motion of the a -th M2-brane in the B -th complex plane in which the corresponding “angular momentum” h_a^B is determined by the assignment of the other preserved charges. We have also introduced the fermionic variable $\lambda_{\alpha a} := \psi_{\alpha B a}$ with $A = B$, which turns out to be the superpartner of r_a^C , $C = 1, 2, 3$, as we will see the supersymmetry transformations (3.140) and (3.141).

The action (3.115) leads to the following equations of motion

$$\ddot{x}_a = \frac{[kC_1(E_a) + \sum_{A \neq B} h_a^A + \sum_{A \neq B} \psi^{\dagger \alpha A a} \psi_{\alpha A a} + \lambda^{\dagger \alpha a} \lambda_{\alpha a}]^2}{4x_a^3} \quad (3.116)$$

$$\ddot{r}_a^A = \frac{(h_a^A)^2}{4(r_a^A)^3} \quad (3.117)$$

$$\dot{\psi}_{\alpha A a} = i \frac{kC_1(E_a) + \sum_{A \neq B} h_a^A + \sum_{A \neq B} \psi^{\dagger \alpha A a} \psi_{\alpha A a} + \lambda^{\dagger \alpha a} \lambda_{\alpha a}}{2x_a} \psi_{\alpha A a} \quad (3.118)$$

$$\dot{\psi}^{\dagger \alpha A a} = -i \frac{kC_1(E_a) + \sum_{A \neq B} h_a^A + \sum_{A \neq B} \psi^{\dagger \alpha A a} \psi_{\alpha A a} + \lambda^{\dagger \alpha a} \lambda_{\alpha a}}{2x_a} \psi^{\dagger \alpha A a} \quad (3.119)$$

$$\dot{\lambda}_{\alpha a} = i \frac{kC_1(E_a) + \sum_{A \neq B} h_a^A + \sum_{A \neq B} \psi^{\dagger \alpha A a} \psi_{\alpha A a} + \lambda^{\dagger \alpha a} \lambda_{\alpha a}}{2x_a} \lambda_{\alpha a} \quad (3.120)$$

$$\dot{\lambda}^{\dagger \alpha a} = -i \frac{kC_1(E_a) + \sum_{A \neq B} h_a^A + \sum_{A \neq B} \psi^{\dagger \alpha A a} \psi_{\alpha A a} + \lambda^{\dagger \alpha a} \lambda_{\alpha a}}{2x_a} \lambda^{\dagger \alpha a}. \quad (3.121)$$

Using the fermionic equations of motion (3.118)–(3.121), we can check that the Gauss law constraint (3.114) has no time dependence, i.e. $\dot{\phi}_{0a} = 0$.

The canonical momenta are given by

$$p^a := \frac{\partial L}{\partial \dot{x}_a} = 2\dot{x}^a, \quad P_A^a := \frac{\partial L}{\partial \dot{r}_a^A} = 2\dot{r}_a^A \quad (3.122)$$

$$\pi^{\alpha Aa} := \frac{\vec{\partial} L}{\partial \dot{\psi}_{\alpha Aa}} = \frac{i}{2} \psi^{\dagger \alpha Aa}, \quad \tilde{\pi}_{\alpha Aa} := \frac{\vec{\partial} L}{\partial \dot{\psi}^{\dagger \alpha Aa}} = \frac{i}{2} \psi_{\alpha Aa} \quad (3.123)$$

$$\Pi^{\alpha a} := \frac{\vec{\partial} L}{\partial \dot{\lambda}_{\alpha a}} = \frac{i}{2} \lambda^{\dagger \alpha a}, \quad \tilde{\Pi}_{\alpha a} := \frac{\vec{\partial} L}{\partial \dot{\lambda}^{\dagger \alpha a}} = \frac{i}{2} \lambda_{\alpha a}. \quad (3.124)$$

The fermionic canonical momenta provide the second class constraints

$$\phi_1^{\alpha Aa} := \pi^{\alpha Aa} - \frac{i}{2} \psi^{\dagger \alpha Aa} = 0, \quad \phi_{2\alpha Aa} := \tilde{\pi}_{\alpha Aa} - \frac{i}{2} \psi_{\alpha Aa} = 0 \quad (3.125)$$

$$\phi_3^{\alpha a} := \Pi^{\alpha a} - \frac{i}{2} \lambda^{\dagger \alpha a} = 0, \quad \phi_{4\alpha a} := \tilde{\Pi}_{\alpha a} - \frac{i}{2} \lambda_{\alpha a} = 0. \quad (3.126)$$

Taking account into the constraints (3.125) and (3.126), we find the Dirac brackets

$$[x_a, p^b]_{DB} = \delta_{ab}, \quad [r_a^A, P_B^b]_{DB} = \delta_{AB} \delta_{ab} \quad (3.127)$$

$$[\psi_{\alpha Aa}, \psi^{\dagger \beta Bb}]_{DB} = i \delta_{\alpha\beta} \delta_{AB} \delta_{ab}, \quad [\lambda_{\alpha a}, \lambda^{\dagger \beta b}]_{DB} = i \delta_{\alpha\beta} \delta_{ab}. \quad (3.128)$$

The action (3.115) possesses the one-dimensional conformal invariance

$$\delta t = f(t) = a + bt + ct^2, \quad \delta \partial_0 = -\dot{f} \partial_0 \quad (3.129)$$

$$\delta x_a = \frac{1}{2} \dot{f} x_a, \quad \delta r_a^A = \frac{1}{2} \dot{f} r_a^A \quad (3.130)$$

$$\delta \psi_{\alpha Aa} = 0, \quad \delta \psi_a^{\dagger \alpha A} = 0 \quad (3.131)$$

$$\delta \lambda_{\alpha a} = 0, \quad \delta \lambda_a^{\dagger \alpha} = 0. \quad (3.132)$$

Using the Noether's procedure we find the $SL(2, \mathbb{R})$ generators

$$H = \sum_{a=1}^N \left[\frac{p_a^2}{4} + \frac{(kC_1(E_a) + \sum_{A \neq B} h_a^A + \sum_A \psi^{\dagger \alpha Aa} \psi_{\alpha Aa} + \lambda^{\dagger \alpha a} \lambda_{\alpha a})^2}{4x_a^2} \right. \\ \left. + \sum_{A \neq B} \left(\frac{(P_a^A)^2}{4} + \frac{(h_a^A)^2}{4(r_a^A)^2} \right) \right] \quad (3.133)$$

$$D = -\frac{1}{4} \sum_{a=1}^N \left[\{x_a, p_a\} + \sum_{A \neq B} \{r_a^A, P_a^A\} \right] \quad (3.134)$$

$$K = \sum_{a=1}^N \left[x_a^2 + \sum_{A \neq B} (r_a^A)^2 \right]. \quad (3.135)$$

Here we have absorbed the time dependent part of D and K by similarity transformations (3.43).

Also the action (3.115) is invariant under the following fermionic transformations

$$\delta x_a = \frac{i}{\sqrt{2}} (\epsilon^{\alpha C} \psi_{\alpha C a} + \epsilon_{\alpha C}^\dagger \psi_a^{\dagger \alpha C}) \quad (3.136)$$

$$\delta r_a^C = \frac{i}{2} [(\omega^{\alpha C D} \psi_{\alpha D a}) e^{-i\theta_a^C} + (\psi_a^{\dagger \alpha D} \omega_{\alpha C D}) e^{i\theta_a^C} - (\epsilon^{\alpha C} \lambda_{\alpha a}) e^{-i\theta_a^C} - (\epsilon_{\alpha C}^\dagger \lambda_a^{\dagger \alpha}) e^{i\theta_a^C}] \quad (3.137)$$

$$\begin{aligned} \delta \psi_{\alpha C a} = & \left(\dot{r}_a^D + i \frac{h_a^D}{2r_a^D} \right) e^{i\theta_a^D} \omega_{\alpha C D} \\ & + \sqrt{2} \left(\dot{x}_a - i \frac{kC_1(E_a) + \sum_{D \neq B} h_a^D + \psi^{\dagger \alpha D a} \psi_{\alpha D a} + \lambda^{\dagger \alpha a} \lambda_{\alpha a}}{2x_a} \right) \epsilon_{\alpha C}^\dagger \\ & - \frac{i}{\sqrt{2}} \frac{l_a}{x_a} \psi_{\alpha C a} \end{aligned} \quad (3.138)$$

$$\begin{aligned} \delta \psi_a^{\dagger \alpha C} = & - \left(\dot{r}_a^D - i \frac{h_a^D}{2r_a^D} \right) e^{-i\theta_a^D} \omega_{\alpha C}^D \\ & + \sqrt{2} \left(\dot{x}_a + i \frac{kC_1(E_a) + \sum_{D \neq B} h_a^D + \psi^{\dagger \alpha D a} \psi_{\alpha D a} + \lambda^{\dagger \alpha a} \lambda_{\alpha a}}{2x_a} \right) \epsilon^{\alpha C} \\ & + \frac{i}{\sqrt{2}} \frac{l_a}{x_a} \psi_a^{\dagger \alpha C} \end{aligned} \quad (3.139)$$

$$\delta \lambda_{\alpha a} = -\epsilon_{\alpha C}^\dagger \left(\dot{r}_a^C + i \frac{h_a^C}{2r_a^C} \right) e^{i\theta_a^C} \quad (3.140)$$

$$\delta \lambda_a^{\dagger \alpha} = - \left(\dot{r}_a^C - i \frac{h_a^C}{2r_a^C} \right) e^{-i\theta_a^C} \epsilon^{\alpha C} \quad (3.141)$$

with $C, D = 1, 2, 3$. Here $\epsilon^{\alpha C}$ and their Hermitian conjugate $\epsilon_{\alpha C}^\dagger$ are infinitesimal fermionic parameters and we have defined

$$\theta_a^C(t) = h_a^C \int^t \frac{dt'}{(r_a^C(t'))^2} \quad (3.142)$$

$$l_a = \epsilon \psi_a - \epsilon^\dagger \psi_a^\dagger. \quad (3.143)$$

3.2.3. $SU(1, 1|6)$ superconformal mechanics

The fact that the transformations (3.136)–(3.141) involve the non-local quantities suggests that there may exist infinitely many conserved non-local charges. However, we see from (3.133) that the Hamiltonian describing the motion in the B -th complex plane associated with the variable x_a and the local charges commute with the others associated with the variables r_a^C 's and the non-local charges. Therefore they are decoupled with one another and we thus can analyze the dynamics in the B -th direction separately. As in Subsection 3.1.3, it is convenient to assign the conserved charges h_a^A and $\lambda^{\dagger \alpha a} \lambda_{\alpha a}$ to be zeros. Then the low-energy dynamics in the B -th complex plane is described by the action

$$S = \int_{\mathbb{R}} dt \sum_{a=1}^N \left[\dot{x}_a^2 - i \psi^{\dagger \alpha A a} \dot{\psi}_{\alpha A a} - \frac{(kC_1(E_a) + \psi^{\dagger \alpha A a} \psi_{\alpha A a})^2}{4x_a^2} \right] \quad (3.144)$$

where $A = 1, 2, 3$ denote the R-symmetry indices. Note that the action (3.144) has the same structure argued in [16–18] for $\mathcal{N} > 4$ superconformal quantum mechanics.

The action (3.144) has the invariance under the $\mathcal{N} = 12$ supersymmetry transformation laws

$$\delta x_a = \frac{i}{\sqrt{2}} (\epsilon^{\alpha A} \psi_{Aa}^\alpha + \epsilon_{\alpha A}^\dagger \psi_a^{\dagger \alpha A}) \quad (3.145)$$

$$\delta \psi_{\alpha Aa} = \sqrt{2} \left(\dot{x}_a - i \frac{g_a}{2x_a} \right) \epsilon_{\alpha A}^\dagger - \frac{i}{\sqrt{2}} \frac{l_a}{x_a} \psi_{\alpha Aa} \quad (3.146)$$

$$\delta \psi_a^{\dagger \alpha A} = \sqrt{2} \left(\dot{x}_a + i \frac{g_a}{2x_a} \right) \epsilon^{\alpha A} + \frac{i}{\sqrt{2}} \frac{l_a}{x_a} \psi_a^{\dagger \alpha A} \quad (3.147)$$

where

$$g_a = kC_1(E_a) + \psi^{\dagger \alpha Aa} \psi_{\alpha Aa}. \quad (3.148)$$

The supersymmetry transformations (3.145)–(3.147) are generated by the supercharges

$$Q_{\alpha A} = \frac{i}{\sqrt{2}} \left(p^a - \frac{g_a}{x_a} \right) \psi_{\alpha Aa} \quad (3.149)$$

$$\tilde{Q}^{\alpha A} = \frac{i}{\sqrt{2}} \left(p^a + \frac{g_a}{x_a} \right) \psi_a^{\dagger \alpha A}. \quad (3.150)$$

Also the action (3.144) has the one-dimensional conformal invariance. The corresponding Noether charges are now expressed as

$$H = \sum_{a=1}^N \left[\frac{p_a^2}{4} + \frac{(kC_1(E_a) + \psi^{\dagger \alpha Aa} \psi_{\alpha Aa})^2}{4x_a^2} \right] \quad (3.151)$$

$$D = -\frac{1}{4} \sum_{a=1}^N \{x_a, p^a\} \quad (3.152)$$

$$K = \sum_{a=1}^N x_a^2. \quad (3.153)$$

According to the Dirac brackets (3.127) and (3.128), quantum operators of the canonical coordinates and momenta obey the quantum brackets

$$[x_a, p^b] = i\delta_{ab}, \quad \{\psi_{\alpha Aa}, \psi^{\dagger \beta Bb}\} = -\delta_{\alpha\beta} \delta_{AB} \delta_{ab}. \quad (3.154)$$

Combining the supercharges and the conformal generators, we find the superconformal boost generators

$$S_{\alpha A} = \sqrt{2}i \sum_a x_a \psi_{\alpha Aa} \quad (3.155)$$

$$\tilde{S}^{\alpha A} = \sqrt{2}i \sum_a x_a \psi_a^{\dagger \alpha A}. \quad (3.156)$$

The R-symmetry generator is given by

$$(J_{\alpha\beta})_{AB} = i \sum_a \psi_a^{\dagger \beta B} \psi_{\alpha Aa}. \quad (3.157)$$

Note that (3.157) is a complex 6×6 matrix with $\alpha, \beta = +, -$ and $A, B = 1, 2, 3$ and it contains 36 complex valued elements.

Under the canonical relations (3.154), the generators form the following algebra

$$[H, D] = iH, \quad [K, D] = -iK, \quad [H, K] = 2iD \quad (3.158)$$

$$[(J_{\alpha\beta})_{AB}, H] = 0, \quad [(J_{\alpha\beta})_{AB}, D] = 0, \quad [(J_{\alpha\beta})_{AB}, K] = 0 \quad (3.159)$$

$$[(J_{\alpha\beta})_{AB}, (J_{\gamma\delta})_{CD}] = i(J_{\alpha\delta})_{AD}\delta_{\beta\gamma}\delta_{BC} - i(J_{\gamma\beta})_{CB}\delta_{\alpha\delta}\delta_{AD} \quad (3.160)$$

$$[H, Q_{\alpha A}] = 0, \quad [D, Q_{\alpha A}] = -\frac{i}{2}Q_{\alpha A}, \quad [K, Q_{\alpha A}] = iS_{\alpha A} \quad (3.161)$$

$$[H, \tilde{Q}^{\alpha A}] = 0, \quad [D, \tilde{Q}^{\alpha A}] = -\frac{i}{2}\tilde{Q}^{\alpha A}, \quad [K, \tilde{Q}^{\alpha A}] = i\tilde{S}^{\alpha A}$$

$$[H, S_{\alpha A}] = -iQ_{\alpha A}, \quad [D, S_{\alpha A}] = \frac{i}{2}S_{\alpha A}, \quad [K, S_{\alpha A}] = 0 \quad (3.162)$$

$$[H, \tilde{S}^{\alpha A}] = -i\tilde{Q}^{\alpha A}, \quad [D, \tilde{S}^{\alpha A}] = \frac{i}{2}\tilde{S}^{\alpha A}, \quad [K, \tilde{S}^{\alpha A}] = 0$$

$$\{Q_{\alpha A}, \tilde{Q}^{\beta B}\} = 2H\delta_{\alpha\beta}\delta_{AB}$$

$$\{S_{\alpha A}, \tilde{S}^{\beta B}\} = 2K\delta_{\alpha\beta}\delta_{AB}$$

$$\{Q_{\alpha A}, \tilde{S}^{\beta B}\} = -2D\delta_{\alpha\beta}\delta_{AB} - 2(J_{\alpha\beta})_{AB} + \frac{i}{2}\left(2\sum_a g_a + 1\right)\delta_{\alpha\beta}\delta_{AB}$$

$$\{\tilde{Q}^{\alpha A}, S_{\beta B}\} = -2D\delta_{\alpha\beta}\delta_{AB} - 2(J_{\alpha\beta}^\dagger)_{AB} - \frac{i}{2}\left(2\sum_a g_a + 1\right)\delta_{\alpha\beta}\delta_{AB} \quad (3.163)$$

$$[(J_{\alpha\beta})_{AB}, Q_{\gamma C}] = iQ_{\alpha A}\delta_{\beta\gamma}\delta_{BC}, \quad [(J_{\alpha\beta})_{AB}, S_{\gamma, C}] = iS_{\alpha A}\delta_{\beta\gamma}\delta_{BC}$$

$$[(J_{\alpha\beta})_{AB}, \tilde{Q}^{\gamma C}] = -i\tilde{Q}^{\alpha A}\delta_{\beta\gamma}\delta_{BC}, \quad [(J_{\alpha\beta})_{AB}, \tilde{S}^{\gamma, C}] = -i\tilde{S}^{\alpha A}\delta_{\beta\gamma}\delta_{BC}. \quad (3.164)$$

The Hamiltonian H , the dilatation generator D and the conformal boost generator form the one-dimensional conformal algebra $\mathfrak{so}(1, 2) = \mathfrak{sl}(2, \mathbb{R}) = \mathfrak{su}(1, 1)$. As each of the supercharges $Q_{\alpha A}$ and $\tilde{Q}^{\alpha A} = -(Q_{\alpha A})^\dagger$ contain six real components, there exist twelve supercharges. They are the square roots of the Hamiltonian H . In addition, there are as many superconformal charges $S_{\alpha A}$ and $\tilde{S}^{\alpha A}$, which are the square roots of the conformal boost generator K . The anti-commutators of the fermionic charges generate an extra bosonic R-symmetry generators $(J_{\alpha\beta})_{AB}$. They form the $\mathfrak{u}(6)$ algebra (3.160). Thus the action (3.144) describes the $SU(1, 1|6)$ invariant $\mathcal{N} = 12$ superconformal mechanics. In fact this belongs to the list of the simple supergroup for superconformal quantum mechanics [33,34].

4. Curved M2-branes and topological twisting

4.1. M2-branes wrapping a holomorphic curve

The BLG action (2.15) and the ABJM action (2.26) may describe the dynamics of probe membranes propagating in a fixed background geometry with an $SO(8)$ and an $SU(4)$ holonomy respectively. For both cases, the world-volume M_3 is considered as a flat space-time $\mathbb{R}^{1,2}$ or

$\mathbb{R} \times T^2$. Now let us consider more general situations where curved M2-branes reside in some fixed curved background geometries. If we naively put the theory on a general three-dimensional manifold, all supersymmetries are broken. However, here we shall wrap the M2-branes on a Riemann surface Σ_g of genus g that preserves supersymmetry (i.e. supersymmetric two-cycles) as the form

$$M_3 = \mathbb{R} \times (\Sigma_g \subset X) \quad (4.1)$$

where \mathbb{R} is viewed as a time direction and X is a real $2(n+1)$ -dimensional space preserving supersymmetry with vanishing three-form gauge field. The only known supersymmetric two-cycles, i.e. calibrated two-cycles in special holonomy backgrounds are holomorphic curves in Calabi–Yau spaces and the corresponding two-form calibrations are Kähler calibrations. So we take the ambient space X as an $(n+1)$ -dimensional Calabi–Yau space and the other space as flat. The geometry of the M-theory is of the form

$$\mathbb{R}^{1,8-2n} \times CY_{n+1}. \quad (4.2)$$

4.1.1. Supersymmetry in Calabi–Yau space

In order to count the number of preserved supersymmetries in our setup, we firstly need to know the dimension of the vector space formed by the corresponding Killing spinor ϵ , that is the amount of supersymmetries in the background geometry. Since we are now considering the background geometries with vanishing four-form flux, the Killing spinor equation is given by

$$\nabla_M \epsilon = \left(\partial_M + \frac{1}{4} \omega_{MPQ} \Gamma^{PQ} \right) \epsilon = 0 \quad (4.3)$$

where ω_{MPQ} , $M, N, P, Q = 0, 1, \dots, 10$, is an eleven-dimensional Levi-Civita spin connection. This leads to the integrability condition

$$[\nabla_M, \nabla_N] \epsilon = \frac{1}{4} R_{MNPQ} \Gamma^{PQ} \epsilon = 0. \quad (4.4)$$

Eq. (4.4) implies that a Killing spinor ϵ transforms as a singlet under the restricted holonomy group $H \subset Spin(1, 10)$ generated by $R_{MNPQ} \Gamma^{PQ}$. In other words, the amount of preserved supersymmetries in the special holonomy manifold is equivalent to the number of singlets in the decomposition of the spinor representation **32** of $Spin(1, 10)$ into the representation of the holonomy group H . In our case the background geometries are taken as Calabi–Yau $(n+1)$ -folds with the holonomy $H = SU(n+1)$, $n = 1, 2, 3, 4$, and the decompositions are as follows.

1. CY_5

In this case the geometry is of the form $\mathbb{R} \times CY_5$. This splits the $Spin(10)$ into $SU(5)$ and the corresponding decomposition of the spinor representation is given by

$$\begin{aligned} \mathbf{16} &= \mathbf{10}_- \oplus \bar{\mathbf{5}}_3 \oplus \mathbf{1}_{-5} \\ \mathbf{16}' &= \mathbf{10}_+ \oplus \mathbf{5}_{-3} \oplus \mathbf{1}_5. \end{aligned} \quad (4.5)$$

The existence of two singlets implies that the space $\mathbb{R} \times CY_5$ preserves $\frac{2}{32} = \frac{1}{16}$ supersymmetries.

Let us define an explicit set of projections defining the Killing spinors. To this end we need to specify how the Calabi–Yau spaces live in the eleven-dimensional space–time. We shall consider the situations where the Calabi–Yau manifolds fill in the order (x^1, x^2) , (x^9, x^{10}) ,

(x^7, x^8) , (x^5, x^6) and (x^3, x^4) . Then the Killing spinors can be defined by the eigenvalues ± 1 for the following set of commuting matrices

$$\Gamma^{12910}, \quad \Gamma^{91078}, \quad \Gamma^{7856}, \quad \Gamma^{5634}. \quad (4.6)$$

The corresponding Killing spinors for CY_5 can be defined by the projection

$$\Gamma^{12910}\epsilon = \Gamma^{91078}\epsilon = \Gamma^{7856}\epsilon = \Gamma^{5634}\epsilon = -\epsilon. \quad (4.7)$$

Note that this implies that $\Gamma^{012}\epsilon = \epsilon$.

2. CY_4

For this case the geometry is the product form $\mathbb{R}^{1,2} \times CY_4$. This leads to the decomposition of the $Spin(8)$ into $SU(4)$ and that of the spinor representation

$$\begin{aligned} \mathbf{8}_s &= \mathbf{6}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2} \\ \mathbf{8}_c &= \mathbf{4}_- \oplus \mathbf{4}_+. \end{aligned} \quad (4.8)$$

We see that the decomposition provides two singlets from sixteen components. Thus the geometry $\mathbb{R}^{1,2} \times CY_4$ can preserve $\frac{2}{16} = \frac{1}{8}$ supersymmetries. In this case the projection for the Killing spinor is given by

$$\Gamma^{12910}\epsilon = \Gamma^{91078}\epsilon = \Gamma^{7856}\epsilon = -\epsilon. \quad (4.9)$$

3. CY_3

In this case the geometry is given by $\mathbb{R}^{1,4} \times CY_3$. This decomposes the $Spin(6)$ into $SU(3)$ and correspondingly spinor representation decomposes as

$$\begin{aligned} \mathbf{4} &= \mathbf{3}_- \oplus \mathbf{1}_3 \\ \bar{\mathbf{4}} &= \mathbf{3}_+ \oplus \mathbf{1}_{-3}. \end{aligned} \quad (4.10)$$

The appearance of two singlets from eight components means that there are $\frac{2}{8} = \frac{1}{4}$ supersymmetries in the product space $\mathbb{R}^{1,4} \times CY_3$. Therefore the Killing spinor can be defined by the projection

$$\Gamma^{12910}\epsilon = \Gamma^{91078}\epsilon = -\epsilon. \quad (4.11)$$

4. $K3$

For this case the geometry is the product space $\mathbb{R}^{1,6} \times K3$. The decomposition of $Spin(4)$ into $SU(2) \times SU(2)$ gives rise to that of the spinor representation

$$\begin{aligned} \mathbf{2} &= (\mathbf{2}, \mathbf{1}) \\ \mathbf{2}' &= (\mathbf{1}, \mathbf{2}). \end{aligned} \quad (4.12)$$

The presence of two singlets under one part of the $SU(2)$ implies that there are $\frac{2}{4} = \frac{1}{2}$ supersymmetries in the geometry $\mathbb{R}^{1,6} \times K3$. The corresponding Killing spinors satisfy the projection

$$\Gamma^{12910}\epsilon = -\epsilon. \quad (4.13)$$

4.1.2. Calibration and supersymmetric cycle

Now consider the situation where the M2-branes wrapping a Riemann surface Σ_g propagate in a Calabi–Yau space without back reaction. To preserve supersymmetry on the world-volume, Σ_g turns out to be a calibrated two-cycle, i.e. holomorphic curve of a Calabi–Yau manifold. To see this let us briefly review the background material concerning a calibration. In general a calibration on a special holonomy manifold X is a differential p -form φ obeying [38]

$$d\varphi = 0 \quad (4.14)$$

$$\varphi|_{\mathcal{C}_p} \leq \text{Vol}|_{\mathcal{C}_p}, \quad \forall \mathcal{C}_p \quad (4.15)$$

where \mathcal{C}_p is any p -cycle in X and Vol is the volume form on the cycle induced from the metric on X . A p -cycle Σ is said to be calibrated by φ if it satisfies

$$\varphi|_{\Sigma} = \text{Vol}|_{\Sigma}. \quad (4.16)$$

We remark that a calibrated submanifold is a minimal surface in their homology class because

$$\text{Vol}(\Sigma) = \int_{\Sigma} \varphi = \int_{M_{p+1}} d\varphi + \int_{\Sigma'} \varphi = \int_{\Sigma'} \varphi \leq \text{Vol}(\Sigma') \quad (4.17)$$

where Σ' is another p -cycle in the same homology class such that $\partial M_{p+1} = \Sigma - \Sigma'$.

It is known that Calabi–Yau $(n+1)$ -folds admit two calibrations; the Kähler form J and the holomorphic $(n+1, 0)$ -form Ω . One can construct calibrations as bilinear forms of spinors [39,40]

$$J_{MN} = i\epsilon^{\dagger} \Gamma_{MN} \epsilon \quad (4.18)$$

$$\Omega_{M_1 \dots M_{n+1}} = \epsilon^T \Gamma_{M_1 \dots M_{2(n+1)}} \epsilon \quad (4.19)$$

Now we consider the condition so that a bosonic configuration of membranes is supersymmetric. Since one can always add a second probe brane without breaking supersymmetry if it is wrapped on the supersymmetric cycle which the original probe brane is wrapping, a simple way to find such condition is to analyze an effective world-volume action of a single membrane [41]. The action for a supermembrane coupled to $d = 11$ supergravity is given by [42]

$$S = \int d^3x \left[\frac{1}{2} \sqrt{-h} h^{\mu\nu} \partial_{\mu} X^M \partial_{\nu} X^N g_{MN} - \frac{1}{2} \sqrt{-h} \right. \\ \left. - i \sqrt{-h} h^{\mu\nu} \bar{\Theta} \Gamma_{\mu} \nabla_{\nu} \Theta + \frac{1}{6} \epsilon^{\mu\nu\lambda} C_{MNP} \partial_{\mu} X^M \partial_{\nu} X^N \partial_{\lambda} X^P \right] \quad (4.20)$$

where $h_{\mu\nu}$, $\mu, \nu = 0, 1, 2$, is the metric of the world-volume, $h = \det(h_{\mu\nu})$, g_{MN} , $M = 0, 1, \dots, 10$, is the $d = 11$ space–time metric. X^M is a space–time coordinate and Θ is a fermionic space–time coordinate. C_{MNP} is a three-form gauge field, which is now taken to be zero in our background geometries. The action (4.20) is invariant under the rigid supersymmetry transformations

$$\delta_{\epsilon} X^M = i\bar{\epsilon} \Gamma^M \Theta \quad (4.21)$$

$$\delta_{\epsilon} \Theta = \epsilon \quad (4.22)$$

where ϵ is a constant anti-commuting eleven-dimensional spinor. Also the action (4.21) has a local fermionic symmetry, called κ -symmetry. The κ -symmetry transformation is given by

$$\delta_\kappa X^M = 2i\bar{\Theta}\Gamma^M P_+ \kappa(x) \quad (4.23)$$

$$\delta_\kappa \Theta = 2P_+ \kappa(x) \quad (4.24)$$

where $\kappa(x)$ is a $d = 11$ spinor and the matrix

$$P_\pm = \frac{1}{2} \left(1 \pm \frac{1}{6\sqrt{-h}} \epsilon^{\mu\nu\lambda} \partial_\mu X^M \partial_\nu X^N \partial_\lambda X^P \Gamma_{MNP} \right) \quad (4.25)$$

is a projection operator satisfying

$$P_\pm^2 = 1, \quad P_+ P_- = 0, \quad P_+ + P_- = 1. \quad (4.26)$$

To extract the physical degrees of freedom, we must choose the suitable gauge that fixes the local world-volume reparametrization and the local κ -symmetry. Let us fix the reparametrization by choosing $x^0 = X^0$. Then the projection operator (4.25) can be expressed as

$$P_\pm = \frac{1}{2} (1 \pm \Gamma) \quad (4.27)$$

where

$$\Gamma := \frac{1}{2\sqrt{\det(h_{\Sigma ij})}} \Gamma^0 \epsilon^{ij} \partial_i X^M \partial_j X^N \Gamma_{MN}. \quad (4.28)$$

Here $h_{\Sigma ij}$, $i, j = 1, 2$, is the metric of the Riemann surface wrapped by the M2-brane and $\sqrt{\det(h_{\Sigma ij})}$ is the area of the surface. As a next step we want to fix the local κ -symmetry on the world-volume. In order for a bosonic world-volume configuration to be supersymmetric, the global supersymmetry transformations (4.22) need to be compensated for by the κ -symmetry transformations (4.24)

$$(\delta_\epsilon + \delta_\kappa)\Theta = \epsilon + 2P_+ \kappa(x) = 0. \quad (4.29)$$

Acting P_- on both sides we find that

$$P_- \epsilon = \frac{1 - \Gamma}{2} \epsilon = 0. \quad (4.30)$$

Therefore the supersymmetry preserved by the M2-branes is given by the Killing spinor ϵ which obeys the projection (4.29). Noting that $\Gamma^2 = 1$ and $\Gamma^\dagger = \Gamma$, we find that

$$\epsilon^\dagger \frac{1 - \Gamma}{2} \epsilon = \epsilon^\dagger \frac{(1 - \Gamma)(1 - \Gamma)}{2} \epsilon = \left| \frac{1 - \Gamma}{\sqrt{2}} \epsilon \right|^2 \geq 0. \quad (4.31)$$

By normalizing the Killing spinors such that $\epsilon^\dagger \epsilon = 1$, the inequality (4.31) can be rewritten as

$$\text{Vol}(\Sigma_g) \geq \varphi \quad (4.32)$$

where $\text{Vol}(\Sigma_g) = \sqrt{\det(h_{\Sigma ij})}$ is the area of the Riemann surface and φ is the differential two-form defined by

$$\varphi = -\frac{1}{2} (\bar{\epsilon} \Gamma_{MN} \epsilon) dX^M \wedge dX^N. \quad (4.33)$$

Hence the two-form (4.33) satisfies the condition (4.15) for the calibration and has the bilinear expression for Kähler calibration J (see (4.18)). Moreover it can be shown that the two-form (4.33) obeys the other required condition (4.14) for the calibration by using the supersymmetry

algebra [43]. Therefore we can conclude that the two-form (4.33) is a Kähler calibration and that the supersymmetric two-cycle Σ_g wrapped by the M2-branes is a calibrated two-cycle, i.e. a holomorphic curve. Notice that (4.29) is precisely the chirality condition $\Gamma^{012}\epsilon = 0$ imposed on the supersymmetry parameters in the BLG-model (see (2.22)).

At this stage we are ready to count the number of preserved supersymmetries in our M2-brane configurations by combining the two different types of projections; the projections (4.7), (4.9), (4.11) and (4.13) for the background Calabi–Yau manifolds and the projection (4.29) (or (2.22)) for the membranes wrapped around a calibrated two-cycle Σ_g . In most of the cases wrapped branes break half of the supersymmetries preserved by the special holonomy manifolds according to the additional projection for the branes wrapping calibrated cycles. However, for the Calabi–Yau 5-fold the projection condition (4.29) for the M2-branes does not give rise to a further constraint on the surviving two Killing spinors. This implies that M2-branes can wrap a holomorphic curve in a Calabi–Yau 5-fold without breaking down the supersymmetry. The amounts of preserved supersymmetries by the M2-branes wrapping holomorphic curves Σ_g in Calabi–Yau spaces are summarized as

$$\mathcal{N} = \begin{cases} 8 & \text{for } \Sigma_g \subset K3 \\ 4 & \text{for } \Sigma_g \subset CY_3 \\ 2 & \text{for } \Sigma_g \subset CY_4 \\ 2 & \text{for } \Sigma_g \subset CY_5. \end{cases} \quad (4.34)$$

Upon the dimensional reduction to \mathbb{R} , the arising quantum mechanics on \mathbb{R} will have the same number of supersymmetries.

4.2. Topological twisting

In general a quantum field theory on the curved M2-branes interacts with gravity, however, it is also possible to get a supersymmetric quantum field theory on $\mathbb{R} \times \Sigma_g$ by taking the appropriate decoupling limit $l_p \rightarrow 0$ while keeping the volume of Σ_g and that of X fixed. In order to derive such low-energy effective theories on the curved world-volume, we recall how the BLG-model describes the dynamics of the flat M2-branes. In the BLG-model the fields and supercharges transform under $SO(2)_E \times SO(8)_R$ as

$$\begin{aligned} X_a^I &: \mathbf{8}_{v0} \\ \Psi_a &: \mathbf{8}_{c+} \oplus \mathbf{8}_{c-} \\ \epsilon &: \mathbf{8}_{s+} \oplus \mathbf{8}_{s-}. \end{aligned} \quad (4.35)$$

The eight scalar fields X^I 's transform as the vector representations of the R-symmetry $SO(8)_R$ which represents the rotational group of the transverse space of the M2-branes. In other words, they are sections of the normal bundle, which is trivial in this case. However, corresponding to the geometry given in (4.1), now the tangent bundle T_X of the ambient Calabi–Yau manifold X is decomposed as

$$T_X = T_\Sigma \oplus N_\Sigma \quad (4.36)$$

where T_Σ is the tangent bundle over the Riemann surface Σ_g and N_Σ is the normal bundle over the surface. Therefore we need to take into account the existence of the non-trivial normal bundle of calibrated cycles and to introduce new dynamical variables instead of the original scalar fields. These transitions from scalars, i.e. trivial normal bundle to the non-trivial normal

bundles are intimately connected with the way in which the field theory on $\mathbb{R} \times \Sigma_g$ realizes supersymmetry. Along with the coupling to the curvature on the Riemann surface, there now exists a coupling to an external $SO(2n)$ gauge group, the R-symmetry background. Thus one can preserve supersymmetry on the holomorphic Riemann surface by choosing the $SO(2)$ Abelian background from the $SO(2n)$ appropriately.

There is a beautiful observation that such an effective description for curved branes can be obtained by topological twisting [10]. Here we attempt to twist the BLG-model to obtain the low-energy descriptions for the curved M2-branes.⁵

Schematically topological twisting procedure can be achieved by replacing the original Euclidean rotational group $SO(2)_E$ on the Riemann surface by a different subgroup $SO(2)'_E$ of $SO(2)_E \times SO(8)_R$. Although there are many possible ways to pick such subgroups, here we will consider the following decomposition

$$\begin{aligned} SO(8) &\supset SO(8-2n) \times SO(2n) \\ &\supset SO(8-2n) \times SO(2)_1 \times \cdots \times SO(2)_n. \end{aligned} \quad (4.37)$$

The $SO(8-2n)$ is a rotational group of the Euclidean space perpendicular to the Riemann surface, while the $SO(2)_i$ are diagonal subgroups of the external $SO(2n)$ gauge group. The meaning of this decomposition is that the Calabi–Yau manifold X enjoys the decomposable line bundles as the form

$$X = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n \rightarrow \Sigma_g. \quad (4.38)$$

Under the decomposition (4.37), the R-charges for $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$ are determined as follows:

1. $SO(8) \supset SO(6) \times SO(2)_1$

$$\begin{aligned} \mathbf{8}_v &= \mathbf{6}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2} \\ \mathbf{8}_s &= \mathbf{4}_+ \oplus \bar{\mathbf{4}}_- \\ \mathbf{8}_c &= \mathbf{4}_- \oplus \bar{\mathbf{4}}_+. \end{aligned} \quad (4.39)$$

2. $SO(8) \supset SO(4) \times SO(2)_1 \times SO(2)_2$

$$\begin{aligned} \mathbf{8}_v &= \mathbf{4}_{00} \oplus \mathbf{1}_{02} \oplus \mathbf{1}_{0-2} \oplus \mathbf{1}_{20} \oplus \mathbf{1}_{-20} \\ \mathbf{8}_s &= \mathbf{2}_{++} \oplus \mathbf{2}'_{+-} \oplus \mathbf{2}_{--} \oplus \mathbf{2}'_{-+} \\ \mathbf{8}_c &= \mathbf{2}_{-+} \oplus \mathbf{2}'_{--} \oplus \mathbf{2}_{+-} \oplus \mathbf{2}'_{++}. \end{aligned} \quad (4.40)$$

3. $SO(8) \supset SO(2) \times SO(2)_1 \times SO(2)_2 \times SO(2)_3$

$$\begin{aligned} \mathbf{8}_v &= \mathbf{2}_{000} \oplus \mathbf{1}_{002} \oplus \mathbf{1}_{00-2} \oplus \mathbf{1}_{020} \oplus \mathbf{1}_{0-20} \oplus \mathbf{1}_{200} \oplus \mathbf{1}_{-200} \\ \mathbf{8}_s &= \mathbf{1}_{+++} \oplus \mathbf{1}_{++-} \oplus \mathbf{1}_{+-+} \oplus \mathbf{1}_{+--} \oplus \mathbf{1}_{-++} \oplus \mathbf{1}_{-+-} \oplus \mathbf{1}_{--+} \oplus \mathbf{1}_{---} \\ \mathbf{8}_c &= \mathbf{1}_{--+} \oplus \mathbf{1}_{-+-} \oplus \mathbf{1}_{--+} \oplus \mathbf{1}_{-++} \oplus \mathbf{1}_{+-+} \oplus \mathbf{1}_{+--} \oplus \mathbf{1}_{++-} \oplus \mathbf{1}_{+++}. \end{aligned} \quad (4.41)$$

⁵ For the ABJM-model the geometric meaning of the topological twisting is less clear because the classical $SU(4)_R$ R-symmetry reflects the orbifolds. In this paper we will focus on the BLG-model.

4. $SO(8) \supset SO(2)_1 \times SO(2)_2 \times SO(2)_3 \times SO(2)_4$

$$\begin{aligned}
 \mathbf{8}_v &= \mathbf{1}_{0002} \oplus \mathbf{1}_{000-2} \oplus \mathbf{1}_{0020} \oplus \mathbf{1}_{00-20} \oplus \mathbf{1}_{0200} \oplus \mathbf{1}_{0-200} \oplus \mathbf{1}_{2000} \oplus \mathbf{1}_{-2000} \\
 \mathbf{8}_s &= \mathbf{1}_{++++} \oplus \mathbf{1}_{++--} \oplus \mathbf{1}_{+--+} \oplus \mathbf{1}_{-++-} \oplus \mathbf{1}_{--++} \oplus \mathbf{1}_{----} \oplus \mathbf{1}_{--+-} \oplus \mathbf{1}_{-+-+} \\
 \mathbf{8}_c &= \mathbf{1}_{-+++} \oplus \mathbf{1}_{-+--} \oplus \mathbf{1}_{----+} \oplus \mathbf{1}_{---+-} \oplus \mathbf{1}_{-++-+} \oplus \mathbf{1}_{+-+--} \oplus \mathbf{1}_{+--+--} \oplus \mathbf{1}_{+++-+}.
 \end{aligned} \tag{4.42}$$

With one of the decompositions (4.39)–(4.42), we can now define a new generator s' , i.e. the $SO(2)'_E$ charge by

$$s' := s - \sum_{i=1}^n a_i T_i. \tag{4.43}$$

Here s denotes a generator of the original rotational group $SO(2)_E$, T_i represents a generator of the subgroup $SO(2)_i$ diagonally embedded in the external gauge group $SO(2n)$ and a_i 's are the constant parameters characterizing the twisting procedures. From now on we normalize these charges s' , s and T_i such that they are twice as the usual spin on the Riemann surface. Since a_i 's are related to the degrees of the line bundles \mathcal{L}_i 's as

$$\deg(\mathcal{L}_i) = \begin{cases} 2|g-1|a_i & \text{for } g \neq 0 \\ a_i & \text{for } g = 0 \end{cases} \tag{4.44}$$

and the degrees coincide with the first Chern class, the conditions that X is Calabi–Yau are given by

$$\sum_{i=1}^n a_i = \begin{cases} -1 & \text{for } g = 0 \\ 0 & \text{for } g = 1 \\ 1 & \text{for } g > 1. \end{cases} \tag{4.45}$$

Note that the Calabi–Yau conditions (4.45) simultaneously ensure the existence of the covariant constant spinors in the twisted theories. One can easily check that the topological twists underlying the decompositions (4.39), (4.40), (4.41) and (4.42) preserve 8, 4, 2 and 2 supersymmetries as we expect for K3, CY_3 , CY_4 and CY_5 .

Therefore given the decomposable line bundle structures of the Calabi–Yau manifolds (4.38), we can determine the topological twisting procedure from the two conditions (4.44) and (4.45). For a K3 surface, i.e. for $a_2 = a_3 = a_4 = 0$, the local geometry is $T^*\Sigma_g$ and a single twisting parameter a_1 is uniquely determined by the Calabi–Yau condition up to the orientation. For other Calabi–Yau spaces the Calabi–Yau conditions are not so powerful and there are infinitely many ways of the twisting characterized by a_i , or the degrees of the line bundles.

5. SCQM from M2-branes in a K3 surface

Let us study the membranes wrapping a curved Riemann surface of genus $g > 1$ embedded in a K3 surface. In order to preserve supersymmetry one should carry out the topological twisting utilizing the decomposition (4.39). Requiring the existence of covariant constant spinors, the twisting procedure can be uniquely determined since the external gauge field is nothing but an $SO(2)$ Abelian background in this case. Note that the twisting for $\Sigma_g = \mathbb{P}^1$ can be realized just by the orientation reversal.

Under $SO(2)_E \times SO(8)_R \rightarrow SO(2)'_E \times SO(6)_R$, the twisted field theory with $g > 1$ is characterized by the following representations

$$\begin{aligned} X^I: \quad & \mathbf{8}_{v0} \rightarrow \mathbf{6}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2} \\ \epsilon: \quad & \mathbf{8}_{s+} \oplus \mathbf{8}_{s-} \rightarrow \mathbf{4}_0 \oplus \bar{\mathbf{4}}_2 \oplus \mathbf{4}_{-2} \oplus \bar{\mathbf{4}}_0 \\ \psi: \quad & \mathbf{8}_{c+} \oplus \mathbf{8}_{c-} \rightarrow \mathbf{4}_2 \oplus \bar{\mathbf{4}}_0 \oplus \mathbf{4}_0 \oplus \bar{\mathbf{4}}_{-2}. \end{aligned} \quad (5.1)$$

Therefore the bosonic field content is six scalar fields ϕ^I transforming as $\mathbf{6}_0$ and one-forms Φ_z , $\Phi_{\bar{z}}$ transforming as $\mathbf{1}_2 \oplus \mathbf{1}_{-2}$. The fermionic field content is eight scalar fields ψ , $\tilde{\lambda}$ as $\mathbf{4}_0 \oplus \bar{\mathbf{4}}_0$ and one-forms Ψ_z , $\tilde{\Psi}_{\bar{z}}$ as $\mathbf{4}_2 \oplus \bar{\mathbf{4}}_{-2}$. The supersymmetry parameters are eight scalars ϵ , $\tilde{\epsilon}$ as $\mathbf{4}_0 \oplus \bar{\mathbf{4}}_0$ and one-forms $\tilde{\epsilon}_z$, $\epsilon_{\bar{z}}$ as $\bar{\mathbf{4}}_2 \oplus \mathbf{4}_{-2}$. Here and hereafter we distinguish $\mathbf{4}$ and $\bar{\mathbf{4}}$ in terms of tildes over the fermionic objects.

We should note that there are six bosonic scalar fields and eight fermionic scalar charges in the twisted theory. Since a Riemann surface is a real two-dimensional manifold and there are six scalar fields, the theory should describe the circumstance where the two-cycle lives in a $2 + (8 - 6) = 4$ -dimensional curved manifold X . The existence of eight scalar supercharges indicates that the four-manifold preserves $\frac{8}{16} = \frac{1}{2}$ of the supersymmetries. This is the case where a holomorphic Riemann surface Σ_g is embedded in a K3 surface.

Locally the K3 geometry is the cotangent bundle $T^*\Sigma_g$. The remaining two scalar fields combine to yield one-forms on the Riemann surface. They represent the motion of the M2-branes along the non-trivial normal bundle N_Σ over the Riemann surface inside the K3 surface. Under the $SO(6)$ rotational group of the six uncompactified dimensions, the six scalars transform as vector representations $\mathbf{6}_v$ and the one-forms are just singlets. We take the eleven-dimensional space–time configuration as

	0	1	2	3	4	5	6	7	8	9	10
K3	×	○	○	×	×	×	×	×	×	○	○
M2	○	○	○	×	×	×	×	×	×	×	×
Σ_g	×	○	○	×	×	×	×	×	×	×	×

(5.2)

where \circ denotes the direction in which the geometrical objects extend, while \times denotes the direction in which they localize. Note that the projection (4.13) for the K3 surface encodes the configuration (5.2). The world-volume of the M2-branes extend to a time direction x^0 and spacial directions x^1, x^2 . The spacial directions x^1, x^2 are tangent to the compact Riemann surface in the K3 surface. The normal geometry of the M2-branes is divided into two parts; one is the normal bundle N_Σ inside the K3 surface, extending to two directions x^9, x^{10} and the other is the flat Euclidean space transverse to the K3 surface, labeled by x^3, \dots, x^8 .

5.1. Twisted theory

Firstly our space–time configuration (5.2) breaks down the space–time symmetry group $SO(1, 10)$ to $SO(2)_E \times SO(6)_R \times SO(2)_1$. So the $SO(1, 10)$ gamma matrix can be decomposed as

$$\begin{cases} \Gamma^\mu = \gamma^\mu \otimes \hat{\Gamma}^7 \otimes \sigma_2, & \mu = 0, 1, 2 \\ \Gamma^{I+2} = \mathbb{I}_2 \otimes \hat{\Gamma}^I \otimes \sigma_2, & I = 1, \dots, 6 \\ \Gamma^{i+8} = \mathbb{I}_2 \otimes \mathbb{I}_8 \otimes \gamma^i, & i = 1, 2 \end{cases} \quad (5.3)$$

where $\hat{\Gamma}^I$ is the $SO(6)$ gamma matrix obeying

$$\{\hat{F}^I, \hat{F}^J\} = 2\delta^{IJ}, \quad (\hat{F}^I)^\dagger = \Gamma^I \quad (5.4)$$

$$\hat{F}^7 = -i\hat{F}^{12\dots 6} = \begin{pmatrix} \mathbb{I}_4 & 0 \\ 0 & -\mathbb{I}_4 \end{pmatrix}. \quad (5.5)$$

Similarly the $SO(1, 10)$ charge conjugation matrix \mathcal{C} is expressed as

$$\mathcal{C} = \epsilon \otimes \hat{C} \otimes \epsilon \quad (5.6)$$

where $\epsilon := i\sigma_2$ is introduced as the charge conjugation matrix with the relations

$$\epsilon^T = -\epsilon, \quad \epsilon \gamma^\mu \epsilon^{-1} = -(\gamma^\mu)^T \quad (5.7)$$

while \hat{C} is the $SO(6)$ charge conjugation matrix satisfying

$$\hat{C}^T = -\hat{C}, \quad \hat{C} \hat{F}^I \hat{C}^{-1} = (\hat{F}^I)^T, \quad \hat{C} \hat{F}^7 \hat{C}^{-1} = -(\hat{F}^7)^T. \quad (5.8)$$

Under the decomposition (5.3), the $SO(8)$ chiral matrix becomes

$$\Gamma^{012} = \Gamma^{34\dots 10} = \mathbb{I}_2 \otimes \hat{F}^7 \otimes \sigma_2. \quad (5.9)$$

For the twisted bosonic fields we set

$$\phi^I := X^{I+2} \quad (5.10)$$

$$\Phi_z := \frac{1}{\sqrt{2}}(X^9 - iX^{10}), \quad \Phi_{\bar{z}} := \frac{1}{\sqrt{2}}(X^9 + iX^{10}) \quad (5.11)$$

$$A_z := \frac{1}{\sqrt{2}}(A_1 - iA_2), \quad A_{\bar{z}} := \frac{1}{\sqrt{2}}(A_1 + iA_2) \quad (5.12)$$

where the bosonic scalar fields ϕ^I 's transform as the vector representations $\mathbf{6}_v$ of the $SO(6)$ global symmetry and the indices $I = 1, \dots, 6$ label the flat transverse directions. The bosonic one-forms, Φ_z and $\Phi_{\bar{z}}$ are the $SO(6)$ -singlets and they describe the motion in the normal geometry N_Σ of the Riemann surface inside the K3 surface. These Higgs fields ϕ^I , Φ_z and $\Phi_{\bar{z}}$ are the 3-algebra valued.

Now consider the twisted fermionic objects. Primitively the fermionic fields Ψ are $SL(2, \mathbb{R})$ spinors that transform as the spinor representations $\mathbf{8}_c$ of the $SO(8)_R$ R-symmetry. As seen from (5.1), under the decomposition $Spin(1, 10) \rightarrow Spin(2) \times Spin(6) \times Spin(2)$, fermionic fields Ψ are split into the representations $\mathbf{4}_2$, $\mathbf{\bar{4}}_0$, $\mathbf{4}_0$ and $\mathbf{\bar{4}}_{-2}$, whose component fields are denoted by ψ_z , $\tilde{\lambda}$, ψ and $\tilde{\psi}_{\bar{z}}$ respectively. Accordingly they can be expanded as

$$\Psi_A^{\alpha\beta} = \frac{i}{\sqrt{2}}\psi_A(\gamma_+\epsilon^{-1})^{\alpha\beta} + i\tilde{\psi}_{\bar{z}A}(\gamma^{\bar{z}}\epsilon^{-1})^{\alpha\beta} - \frac{i}{\sqrt{2}}\tilde{\lambda}_A(\gamma_-\epsilon^{-1})^{\alpha\beta} - i\psi_{zA}(\gamma^z\epsilon^{-1})^{\alpha\beta} \quad (5.13)$$

where the three indices α , A and β denote the $SO(2)_E$ spinor, the $SO(6)_R$ spinor and the $SO(2)_1$ spinor respectively. Here we have introduced the matrices γ_\pm , γ^z and $\gamma^{\bar{z}}$ defined by

$$\gamma_+ := \frac{1}{\sqrt{2}}(\mathbb{I}_2 + \sigma_2), \quad \gamma_- := \frac{1}{\sqrt{2}}(\mathbb{I}_2 - \sigma_2) \quad (5.14)$$

$$\gamma^z := \frac{1}{\sqrt{2}}(\gamma^1 + i\gamma^2) = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} \quad (5.15)$$

$$\gamma^{\bar{z}} := \frac{1}{\sqrt{2}}(\gamma^1 - i\gamma^2) = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}. \quad (5.16)$$

As seen from (5.13), the above matrices enable us to carry out the topological twisting, or in other words the identification of the index α with the index β . The matrices γ_+ and $\gamma^{\bar{z}}$ are associated with the conjugate spinor representations $\mathbf{8}_{c-}$ and yield $\mathbf{4}_0$ and $\bar{\mathbf{4}}_{-2}$, while the other pair of matrices γ_- and γ^z are associated with $\mathbf{8}_{c+}$ and give rise to $\mathbf{4}_2$ and $\bar{\mathbf{4}}_0$. Together with the decomposition (5.9) and the chirality condition (2.12) for Ψ , one can check that the expansion (5.13) leads to the relations; $\hat{F}^7 \psi = \psi$, $\hat{F}^7 \tilde{\Psi}_{\bar{z}} = -\tilde{\Psi}_{\bar{z}}$, $\hat{F}^7 \tilde{\lambda} = -\tilde{\lambda}$ and $\hat{F}^7 \Psi_z = \Psi_z$. For the \mathcal{A}_4 algebra all of these fermionic fields are the fundamental representations of the $SO(4)$ gauge group. We define the conjugate of the $SO(6)$ spinors as

$$\bar{\psi} := \psi^T \hat{C}, \quad \bar{\tilde{\lambda}} := \tilde{\lambda}^T \hat{C}, \quad \bar{\Psi}_z := \Psi_z^T \hat{C}, \quad \bar{\tilde{\Psi}}_{\bar{z}} := \tilde{\Psi}_{\bar{z}}^T \hat{C}. \quad (5.17)$$

Likewise, the supersymmetry parameters originally transform as the $SL(2, \mathbb{R})$ spinor representations of the rotational group of the world-volume and $\mathbf{8}_s$ of the $SO(8)$ R-symmetry in the BLG-model, while in the twisted theory they reduce to the four distinct representations $\mathbf{4}_0$, $\bar{\mathbf{4}}_2$, $\mathbf{4}_{-2}$ and $\bar{\mathbf{4}}_0$. Thus we can write supersymmetry parameters as

$$\epsilon_A^{\alpha\beta} = \frac{i}{\sqrt{2}} \tilde{\epsilon}_A (\gamma_+ \epsilon^{-1})^{\alpha\beta} + i \epsilon_{\bar{z}A} (\gamma^{\bar{z}} \epsilon^{-1})^{\alpha\beta} - \frac{i}{\sqrt{2}} \epsilon_A (\gamma_- \epsilon^{-1})^{\alpha\beta} - i \tilde{\epsilon}_{zA} (\gamma^z \epsilon^{-1})^{\alpha\beta}. \quad (5.18)$$

Here again the indices α , A and β label $SO(2)_E$, $SO(6)_R$ and $SO(2)_1$ respectively. ϵ_A and $\tilde{\epsilon}_A$ are fermionic scalars on an arbitrary Riemann surface transforming as $\mathbf{4}_0$ and $\bar{\mathbf{4}}_0$ respectively. The existence of the eight supersymmetric parameters reflects the fact that the effective theory will be endowed with the corresponding eight supercharges.

In terms of the expressions (5.3), (5.10), (5.11), (5.12) and (5.13), we obtain the twisted BLG Lagrangian

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} (D_0 \phi^I, D_0 \phi^I) - (D_z \phi^I, D_{\bar{z}} \phi^I) + (D_0 \Phi_z, D_0 \Phi_{\bar{z}}) - 2(D_z \Phi_{\bar{w}}, D_{\bar{z}} \Phi_w) \\ & + (\bar{\tilde{\lambda}}, D_0 \psi) + (\bar{\Psi}_z, D_0 \tilde{\Psi}_{\bar{z}}) - (\bar{\tilde{\Psi}}_{\bar{z}}, D_0 \Psi_z) - 2i(\bar{\tilde{\Psi}}_{\bar{z}}, D_z \psi) + 2i(\bar{\tilde{\lambda}}, D_{\bar{z}} \Psi_z) \\ & + \frac{i}{2} (\bar{\tilde{\lambda}} \hat{F}^{IJ}, [\phi^I, \phi^J, \psi]) - i(\bar{\tilde{\Psi}}_{\bar{z}} \hat{F}^{IJ}, [\phi^I, \phi^J, \Psi_z]) \\ & + 2i(\bar{\Psi} \hat{F}^I, [\Phi_{\bar{z}}, \phi^I, \Psi_z]) - 2i(\bar{\tilde{\lambda}} \hat{F}^I, [\Phi_z, \phi^I, \tilde{\Psi}_{\bar{z}}]) \\ & + i(\bar{\tilde{\lambda}}, [\Phi_z, \Phi_{\bar{z}}, \psi]) - 2i(\bar{\tilde{\Psi}}_{\bar{w}}, [\Phi_z, \Phi_{\bar{z}}, \Psi_w]) \\ & - \frac{1}{12} ([\phi^I, \phi^J, \phi^K], [\phi^I, \phi^J, \phi^K]) - \frac{1}{2} ([\Phi_z, \phi^I, \phi^J], [\Phi_{\bar{z}}, \phi^I, \phi^J]) \\ & - \frac{1}{2} ([\Phi_z, \Phi_w, \phi^I], [\Phi_{\bar{z}}, \Phi_{\bar{w}}, \phi^I]) - \frac{1}{2} ([\Phi_z, \Phi_{\bar{w}}, \phi^I], [\Phi_{\bar{z}}, \Phi_w, \phi^I]) \\ & + \frac{1}{6} ([\Phi_z, \Phi_w, \Phi_v], [\Phi_{\bar{z}}, \Phi_{\bar{w}}, \Phi_{\bar{v}}]) + \frac{1}{2} ([\Phi_z, \Phi_w, \Phi_{\bar{v}}], [\Phi_{\bar{z}}, \Phi_{\bar{w}}, \Phi_v]) + \mathcal{L}_{\text{TCS}}. \end{aligned} \quad (5.19)$$

Here we have introduced $(\ , \)$ as the trace form on the 3-algebra introduced in (2.4) and we have defined the covariant derivatives $D_z := \frac{1}{\sqrt{2}}(D_1 - iD_2)$ and $D_{\bar{z}} := \frac{1}{\sqrt{2}}(D_1 + iD_2)$.

The corresponding BRST transformations are given by

$$\delta \phi_a^I = i \bar{\tilde{\epsilon}} \hat{F}^I \tilde{\lambda}_a - i \bar{\tilde{\epsilon}} \hat{F}^I \psi_a \quad (5.20)$$

$$\delta \Phi_{za} = -i \bar{\tilde{\epsilon}} \tilde{\Psi}_{za} \quad (5.21)$$

$$\delta\Phi_{\bar{z}a} = -i\tilde{\epsilon}\tilde{\Psi}_{\bar{z}a} \quad (5.22)$$

$$\delta\psi_a = iD_0\phi_a^I\hat{\Gamma}\tilde{\epsilon} - 2D_{\bar{z}}\Phi_{za}\epsilon + \frac{1}{6}[\phi^I, \phi^J, \phi^K]_a\hat{\Gamma}^{IJK}\tilde{\epsilon} + [\Phi_z, \Phi_{\bar{z}}, \phi^I]_a\hat{\Gamma}^I\tilde{\epsilon} \quad (5.23)$$

$$\delta\tilde{\lambda}_a = iD_0\phi_a^I\hat{\Gamma}^I\epsilon - 2D_z\Phi_{\bar{z}a}\tilde{\epsilon} - \frac{1}{6}[\phi^I, \phi^J, \phi^K]_a\hat{\Gamma}^{IJK}\epsilon + [\Phi_z, \Phi_{\bar{z}}, \phi^I]_a\hat{\Gamma}^I\epsilon \quad (5.24)$$

$$\delta\Psi_{za} = -D_z\phi^I\hat{\Gamma}^I\tilde{\epsilon} - iD_0\Phi_{za}\epsilon + \frac{1}{2}[\Phi_z, \phi^I, \phi^J]_a\hat{\Gamma}^{IJ}\epsilon + \frac{1}{3}[\Phi_w, \Phi_{\bar{w}}, \Phi_z]_a\epsilon \quad (5.25)$$

$$\delta\tilde{\Psi}_{\bar{z}a} = D_{\bar{z}}\phi_a^I\hat{\Gamma}^I\epsilon + iD_0\Phi_{\bar{z}a}\tilde{\epsilon} + \frac{1}{2}[\Phi_{\bar{z}}, \phi^I, \phi^J]_a\hat{\Gamma}^{IJ}\tilde{\epsilon} + \frac{1}{3}[\Phi_{\bar{w}}, \Phi_w, \Phi_{\bar{z}}]_a\tilde{\epsilon} \quad (5.26)$$

$$\delta\tilde{A}_{0a}^b = -\tilde{\epsilon}\hat{\Gamma}^I\phi_c^I\psi_d f^{cdb}_a - \tilde{\epsilon}\hat{\Gamma}^I\phi_c^I\tilde{\lambda}_d f^{cdb}_a - 2\tilde{\epsilon}\Phi_{zc}\tilde{\Psi}_{\bar{z}d} f^{cdb}_a + 2\tilde{\epsilon}\Phi_{\bar{z}c}\Psi_{zd} f^{cdb}_a \quad (5.27)$$

$$\delta\tilde{A}_{za}^b = 2i\tilde{\epsilon}\hat{\Gamma}^I\phi_c^I\psi_{zd} f^{cdb}_a + 2i\tilde{\epsilon}\Phi_{zc}\tilde{\lambda}_d f^{cdb}_a \quad (5.28)$$

$$\delta\tilde{A}_{\bar{z}a}^b = -2i\tilde{\epsilon}\hat{\Gamma}^I\phi_c^I\tilde{\Psi}_{\bar{z}d} f^{cdb}_a + 2i\tilde{\epsilon}\Phi_{\bar{z}c}\psi_d f^{cdb}_a. \quad (5.29)$$

5.2. Derivation of quantum mechanics

Now we consider the reduction to a low-energy effective one-dimensional field theory on \mathbb{R} , that is membrane quantum mechanics. As the size of the Riemann surface shrinks, only the light degrees of freedom are relevant. To keep track of them we have to find the static configurations that minimize the energy, that is the zero-energy conditions. We can replace the zero-energy conditions by a set of BPS equations. In addition, we set all the fermionic fields to zero because we are interested in bosonic BPS configurations. Then the BPS equations, which correspond to the vanishing conditions of the BRST transformations (5.23)–(5.26) for the fermionic fields, are

$$D_z\phi^I = 0, \quad D_{\bar{z}}\phi^I = 0 \quad (5.30)$$

$$D_z\Phi_{\bar{z}} = 0, \quad D_{\bar{z}}\Phi_z = 0 \quad (5.31)$$

$$[\phi^I, \phi^J, \phi^K] = 0 \quad (5.32)$$

$$[\Phi_z, \Phi_{\bar{z}}, \phi^I] = 0, \quad [\Phi_z, \phi^I, \phi^J] = 0, \quad [\Phi_{\bar{z}}, \phi^I, \phi^J] = 0 \quad (5.33)$$

$$[\Phi_w, \Phi_{\bar{w}}, \Phi_z] = 0, \quad [\Phi_{\bar{w}}, \Phi_w, \Phi_{\bar{z}}] = 0. \quad (5.34)$$

We first note that according to the algebraic equations (5.32), (5.33) and (5.34), all the bosonic Higgs fields have to lie in the same plane in the $SO(4)$ gauge group. Thus we can write them as

$$\phi^I = (\phi^{I1}, \phi^{I2}, 0, 0)^T, \quad \Phi_z = (\Phi_z^1, \Phi_z^2, 0, 0)^T, \quad \Phi_{\bar{z}} = (\Phi_{\bar{z}}^1, \Phi_{\bar{z}}^2, 0, 0)^T. \quad (5.35)$$

Correspondingly via supersymmetry one can also write the fermionic partners as

$$\psi = (\psi^1, \psi^2, 0, 0)^T, \quad \tilde{\lambda} = (\tilde{\lambda}^1, \tilde{\lambda}^2, 0, 0)^T \quad (5.36)$$

$$\Psi_z = (\Psi_z^1, \Psi_z^2, 0, 0)^T, \quad \tilde{\Psi}_{\bar{z}} = (\tilde{\Psi}_{\bar{z}}^1, \tilde{\Psi}_{\bar{z}}^2, 0, 0)^T. \quad (5.37)$$

The configurations (5.35)–(5.37) generically break the original $SO(4)$ gauge group down to $U(1) \times U(1)$. Taking into account these solutions and the BPS equations (5.30), (5.31) we find that $\tilde{A}_{z3}^1 = \tilde{A}_{z3}^2 = \tilde{A}_{z4}^1 = \tilde{A}_{z4}^2 = 0$. This implies that these components of the gauge field now become massive by the Higgs mechanism. Then we should follow the time evolution for remaining degrees of freedom in the low-energy effective theory.

To achieve this consistently we further need to impose the Gauss law constraint. This requires that the gauge field is flat; $\tilde{F}_{z\bar{z}} = 0$. Recall that we are now considering the case where the

genus of the Riemann surface is greater than one. In that case the generic flat connections are irreducible. As long as we only consider irreducible flat connections, the Laplacian has no zero modes. Accordingly it is not allowed for scalar fields to have non-trivial values and it is required that $\phi^I = 0$.⁶

To sum up, the above set of equations over the compact Riemann surface reduces to

$$\tilde{F}_{z\bar{z}2}^1 = 0 \quad (5.38)$$

$$\partial_{\bar{z}}\Phi_{z1} + \tilde{A}_{z2}^1\Phi_{z2} = 0 \quad (5.39)$$

$$\partial_{\bar{z}}\Phi_{z2} - \tilde{A}_{z2}^1\Phi_{z1} = 0. \quad (5.40)$$

Let us discuss the generic BPS configuration obeying (5.38)–(5.40). Since we are now considering a compact Riemann surface of genus g , there are g holomorphic $(1, 0)$ -forms ω_i , $i = 1, \dots, g$, and g anti-holomorphic $(0, 1)$ -forms $\bar{\omega}_i$. Let us normalize them as

$$\int_{a_i} \omega_j = \delta_{ij}, \quad \int_{b_i} \omega_j = \Omega_{ij} \quad (5.41)$$

with a_i, b_i being canonical homology basis for $H_1(\Sigma_g)$. The matrix Ω is the period matrix of the Riemann surface. It is a $g \times g$ complex symmetric matrix with positive imaginary part. Eq. (5.38) imposes the flatness condition for the $U(1)$ gauge field \tilde{A}_{z2}^1 . The space of the $U(1)$ flat connection on a compact Riemann surface is the torus known as the Jacobi variety denoted by $\text{Jac}(\Sigma_g)$. The flat gauge fields can be expressed in the form [26]

$$\tilde{A}_{z2}^1 = -2\pi \sum_{i,j=1}^g (\Omega - \bar{\Omega})_{ij}^{-1} \Theta^i \omega_j, \quad \tilde{A}_{\bar{z}2}^1 = 2\pi \sum_{i,j=1}^g (\Omega - \bar{\Omega})_{ij}^{-1} \bar{\Theta}^i \bar{\omega}_j \quad (5.42)$$

where $\Theta^i := \zeta^i + \bar{\Omega}_{ij} \xi^j$ represents the complex coordinate of $\text{Jac}(\Sigma_g)$ which characterizes the twists $e^{2\pi i \xi^i}$ and $e^{-2\pi i \zeta^i}$ around the i -th homology cycles a_i and b_i . Notice that $\xi^i \rightarrow \xi^i + m^i$, $\zeta^i \rightarrow \zeta^i + n^i$ for $n^i, m^i \in \mathbb{Z}$ gives rise to the same point on $\text{Jac}(\Sigma_g)$. This implies that $\text{Jac}(\Sigma_g) = \mathbb{C}^g / L_\Omega$ where L_Ω is the lattice generated by $\mathbb{Z}^g + \Omega \mathbb{Z}^g$. We define a function

$$\varphi := -2\pi \sum_{i,j=1}^g (\Omega - \bar{\Omega})_{ij}^{-1} (\Theta^i f_j(z) - \bar{\Theta}^i \bar{f}_j(\bar{z})) \quad (5.43)$$

where $f_i(z) := \int^z \omega_i$ is the holomorphic function of z that obeys the relations $f_i|_{a_j} = \delta_{ij}$ and $f_i|_{b_j} = \Omega_{ij}$. Then we can write the flat gauge fields as

$$\tilde{A}_{z2}^1 = \partial_z \varphi, \quad \tilde{A}_{\bar{z}2}^1 = \partial_{\bar{z}} \varphi. \quad (5.44)$$

Using the above expressions for the $U(1)$ flat connection, the generic solutions to Eqs. (5.39) and (5.40) can be expressed as

⁶ Such BPS solutions with the irreducible connections have been considered in the four-dimensional topologically twisted Yang–Mills theories defined on the product of two Riemann surfaces [44–46] and the corresponding decoupling limit for the brane description has been argued in [47].

$$\begin{aligned}
\Phi_{z1}(z, \bar{z}) - i\Phi_{z2}(z, \bar{z}) &= e^{-i\varphi(z, \bar{z})} \sum_{i=1}^g x_A^i \omega_i \\
\Phi_{z1}(z, \bar{z}) + i\Phi_{z2}(z, \bar{z}) &= e^{i\varphi(z, \bar{z})} \sum_{i=1}^g x_B^i \omega_i
\end{aligned} \tag{5.45}$$

where $x_A^i, x_B^i \in \mathbb{C}$ are constant on the Riemann surface. Since we take the limit where the Riemann surface Σ_g shrinks to zero size, the space–time configurations of the membranes should be expressed as single-valued functions of z and \bar{z} in the low-energy effective quantum mechanics. In other words, ξ^i and ζ^i can only be integers and therefore the $U(1)$ flat gauge fields \tilde{A}_{z2}^1 and $\tilde{A}_{\bar{z}2}^1$ are quantized. The single-valuedness condition requires that the point of the $\text{Jac}(\Sigma_g)$ is fixed.

Putting all together, the general bosonic BPS configurations are given by

$$\begin{aligned}
\phi^I &= 0 \\
\Phi_z &= \sum_{i=1}^g \begin{pmatrix} \frac{1}{2}(e^{-i\varphi} x_A^i + e^{i\varphi} x_B^i) \\ \frac{i}{2}(e^{-i\varphi} x_A^i - e^{i\varphi} x_B^i) \\ 0 \\ 0 \end{pmatrix} \omega_i, & \Phi_{\bar{z}} &= \sum_{i=1}^g \begin{pmatrix} \frac{1}{2}(e^{i\varphi} \bar{x}_A^i + e^{-i\varphi} \bar{x}_B^i) \\ -\frac{i}{2}(e^{i\varphi} \bar{x}_A^i - e^{-i\varphi} \bar{x}_B^i) \\ 0 \\ 0 \end{pmatrix} \bar{\omega}_i \\
\tilde{A}_z &= \begin{pmatrix} 0 & \partial_z \varphi(z, \bar{z}) & 0 & 0 \\ -\partial_z \varphi(z, \bar{z}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{A}_{z4}^3(z, \bar{z}) \\ 0 & 0 & -\tilde{A}_{z4}^3(z, \bar{z}) & 0 \end{pmatrix}
\end{aligned} \tag{5.46}$$

where \tilde{A}_{z4}^3 and $\tilde{A}_{\bar{z}4}^3$ are the Abelian gauge fields associated with preserved $U(1)$ symmetry and they do not receive any constraints from the BPS conditions.

By virtue of the supersymmetry we can write the corresponding fermionic fields from the bosonic configurations (5.46) as

$$\begin{aligned}
\psi &= 0, & \tilde{\lambda} &= 0 \\
\Psi_z &= \sum_{i=1}^g \begin{pmatrix} \frac{1}{2}(\Psi_A^i + \Psi_B^i) \\ \frac{i}{2}(\Psi_A^i - \Psi_B^i) \\ 0 \\ 0 \end{pmatrix} \omega_i, & \tilde{\Psi}_{\bar{z}} &= \sum_{i=1}^g \begin{pmatrix} \frac{1}{2}(\tilde{\Psi}_A^i + \tilde{\Psi}_B^i) \\ -\frac{i}{2}(\tilde{\Psi}_A^i - \tilde{\Psi}_B^i) \\ 0 \\ 0 \end{pmatrix} \bar{\omega}_i.
\end{aligned} \tag{5.47}$$

Substituting the BPS configuration (5.46) and (5.47) into the twisted action (5.19), we find

$$\begin{aligned}
S &= \int_{\mathbb{R}} dt \int_{\Sigma_g} d^2 z \left[(D_0 \Phi_z^a, D_0 \Phi_{\bar{z}a}) + (\bar{\Psi}_z^a, D_0 \tilde{\Psi}_{\bar{z}a}) - (\tilde{\Psi}_z^a, D_0 \Psi_{za}) \right. \\
&\quad \left. - \frac{k}{2\pi} \tilde{A}_{02}^1 \tilde{F}_{z\bar{z}4}^3 - \frac{k}{4\pi} (\tilde{A}_{z2}^1 \dot{\tilde{A}}_{\bar{z}4}^3 - \tilde{A}_{\bar{z}2}^1 \dot{\tilde{A}}_{z4}^3) \right].
\end{aligned} \tag{5.48}$$

Since the gauge fields $\tilde{A}_{z2}^1, \tilde{A}_{\bar{z}2}^1$ are quantized and their time derivatives do not show up in the effective action, they can be integrated out as the auxiliary fields. They give rise to the constraints $\dot{\tilde{A}}_{z4}^3 = \dot{\tilde{A}}_{\bar{z}4}^3 = 0$.

Making use of the Riemann bilinear relation [48]

$$\int_{\Sigma_g} \omega \wedge \eta = \sum_{i=1}^g \left[\int_{a_i} \omega \int_{b_i} \eta - \int_{b_i} \omega \int_{a_i} \eta \right] \quad (5.49)$$

and performing the integration on Σ_g we obtain the low-energy effective gauged quantum mechanics

$$S = \int_{\mathbb{R}} dt \left[\sum_{i,j} (\text{Im } \Omega)_{ij} (D_0 x^{ia} D_0 \bar{x}_a^j + \bar{\Psi}^{ia} D_0 \tilde{\Psi}_a^j - \bar{\tilde{\Psi}}^{ia} D_0 \Psi_a^j) - k C_1(E) \tilde{A}_{02}^1 \right]. \quad (5.50)$$

Here the indices $a = A, B$ stand for the two internal degrees of freedom for the two M2-branes. The covariant derivatives are defined by

$$D_0 x_A^i = \dot{x}_A^i + i \tilde{A}_{02}^1 x_A^i, \quad D_0 x_B^i = \dot{x}_B^i - i \tilde{A}_{02}^1 x_B^i \quad (5.51)$$

$$D_0 \Psi_A^i = \dot{\Psi}_A^i + i \tilde{A}_{02}^1 \Psi_A^i, \quad D_0 \Psi_B^i = \dot{\Psi}_B^i - i \tilde{A}_{02}^1 \Psi_B^i \quad (5.52)$$

$$D_0 \tilde{\Psi}_A^i = \dot{\tilde{\Psi}}_A^i - i \tilde{A}_{02}^1 \tilde{\Psi}_A^i, \quad D_0 \tilde{\Psi}_B^i = \dot{\tilde{\Psi}}_B^i + i \tilde{A}_{02}^1 \tilde{\Psi}_B^i \quad (5.53)$$

and the Chern number $C_1(E) \in \mathbb{Z}$ is associated to the $U(1)$ principal bundle $E \rightarrow \Sigma_g$ over the Riemann surface

$$C_1(E) = \int_{\Sigma_g} c_1(E) = \frac{1}{2\pi} \int_{\Sigma_g} d^2 z \tilde{F}_{z\bar{z}}^3. \quad (5.54)$$

The action (5.50) has the invariance under the one-dimensional $SL(2, \mathbb{R})$ conformal transformations

$$\delta t = f(t) = a + bt + ct^2, \quad \delta \partial_0 = -\dot{f} \partial_0 \quad (5.55)$$

$$\delta x_a^i = \frac{1}{2} \dot{f} x_a^i, \quad \delta \tilde{A}_{02}^1 = -\dot{f} \tilde{A}_{02}^1 \quad (5.56)$$

$$\delta \Psi_a^i = 0, \quad \delta \tilde{\Psi}_a^i = 0. \quad (5.57)$$

Also the action (5.50) is invariant under the $\mathcal{N} = 8$ supersymmetry transformations

$$\delta x_a^i = 2i \bar{\epsilon} \tilde{\Psi}_a^i, \quad \delta \bar{x}_a^i = 2i \bar{\epsilon} \tilde{\Psi}_a^i \quad (5.58)$$

$$\delta \Psi_a^i = -i D_0 x_a^i \epsilon, \quad \delta \tilde{\Psi}_a^i = i D_0 \bar{x}_a^i \tilde{\epsilon} \quad (5.59)$$

$$\delta \tilde{A}_{02}^1 = 0. \quad (5.60)$$

Therefore we conclude that the $\mathcal{N} = 8$ superconformal gauged quantum mechanics (5.50) may describe the low-energy effective motion of the two wrapped M2-branes around Σ_g probing a K3 surface.

As seen from the action (5.50), the $U(1)$ gauge field \tilde{A}_{02}^1 , due to the absence of the kinetic term, is regarded as an auxiliary field. In consequence the gauge field has no contribution to the Hamiltonian. Hence the corresponding gauge symmetry yields an integral of motion as a moment map $\mu : \mathcal{M} \rightarrow \mathfrak{u}(1)^*$ and we can reduce the phase space \mathcal{M} to $\mathcal{M}_c = \mu^{-1}(c)$ by fixing the inverse of the moment map at a point $c \in \mathfrak{u}(1)^*$. Choosing a temporal gauge $\tilde{A}_{02}^1 = 0$, we find the action

$$S = \int_{\mathbb{R}} dt \sum_{i,j} (\text{Im } \Omega)_{ij} (\dot{x}^{ia} \dot{\bar{x}}_a^j + \bar{\Psi}^{ia} \dot{\Psi}_a^j - \bar{\Psi}^{ia} \dot{\Psi}_a^j) \quad (5.61)$$

and the Gauss law constraint

$$\phi_0 := kC_1(E) + i \sum_{i,j} (\text{Im } \Omega)_{ij} [K_{ij} + 2(\bar{\Psi}_A^i \tilde{\Psi}_A^j - \bar{\Psi}_B^i \tilde{\Psi}_B^j)] = 0 \quad (5.62)$$

where

$$K_{ij} := (\dot{x}_A^i \bar{x}_A^j - x_A^i \dot{\bar{x}}_A^j) - (\dot{x}_B^i \bar{x}_B^j - x_B^i \dot{\bar{x}}_B^j). \quad (5.63)$$

The constraint equation (5.62) requires that all states in the Hilbert space are gauge invariant. In this case the symmetry of the system is not so large as in the previous superconformal gauged quantum mechanical models (3.13) and (3.104). It is curious to know whether the superconformal gauged quantum mechanics (5.50) (or (5.61) together with (5.62)) have a reduced Lagrangian description with an inverse-square type potential. However, our result may drop a hint on the obstructed construction of SCQM that a large class of SCQM could be formulated as “gauged quantum mechanics” with the help of auxiliary gauge fields as in [28–30].

Although it would be helpful to determine the corresponding supermultiplet for our $\mathcal{N} = 8$ superconformal quantum mechanics (5.50), we do not fully understand it because our derivation is not based on the superfield formulation and the reduced quantum mechanical description is missing. Judging from the representations (5.1) of the physical variables under the remaining R-symmetry $SO(6)$, the corresponding supermultiplet may be inferred as the g combinations of $(2, 8, 6)$ multiplet.⁷ However, the elimination of the single auxiliary gauge field \tilde{A}_{02}^1 may reduce the physical degrees of freedom and thus lead to other supermultiplet.

6. Conclusion and discussion

We have studied the IR quantum mechanics resulting from the multiple M2-branes wrapping a compact Riemann surface Σ_g after shrinking the size of the Riemann surface by reducing the BLG-model and the ABJM-model. For $g = 1$ the dimensional reductions of the BLG-model and the ABJM-model yield the low-energy effective $\mathcal{N} = 16$ and $\mathcal{N} = 12$ superconformal gauged quantum mechanical models respectively. After the integration of the auxiliary gauge fields, $OSp(16|2)$ quantum mechanics (3.54) and $SU(1, 1|6)$ quantum mechanics (3.144) emerge from the reduced theories. For $g \neq 1$ the Riemann surface is singled out as a calibrated holomorphic curve in a Calabi–Yau manifold to preserve supersymmetry. The IR quantum mechanical models have $\mathcal{N} = 8, 4, 2$ and 2 supersymmetries for K3, CY_3 , CY_4 and CY_5 respectively. When the Calabi–Yau manifolds are constructed via decomposable line bundles over the Riemann surface, the K3 surface essentially allows for a unique topological twist while for the other Calabi–Yau manifolds there are infinitely many topological twists which are specified by the degrees of the line bundles. In particular we have analyzed the two wrapped M2-branes around a holomorphic genus $g > 1$ curve exploring a K3 surface based on the topologically twisted BLG-model. We have found the $\mathcal{N} = 8$ superconformal gauged quantum mechanics (5.50) that may describe the low-energy dynamics of the wrapped M2-branes in a K3 surface. It is known that [28–30] there are the connections of the gauged quantum mechanics to the conformal mechanical models, the

⁷ Here we use the notation $(n, \mathcal{N}, \mathcal{N} - n)$ with n physical bosonic variables, \mathcal{N} fermions and $\mathcal{N} - n$ auxiliary bosonic variables.

Calogero model and their generalizations. An interesting question is what type of interaction potential, if it exists, may characterize our superconformal gauged quantum mechanics (5.50). This remains open issue for future investigation.

There are a number of future aspects of the present work. In particular, they contain the following impressive subjects:

1. $\text{AdS}_2/\text{CFT}_1$ correspondence

One of the most appealing programs relevant to our work is to attack the $\text{AdS}_2/\text{CFT}_1$ correspondence. This is the most significant case of $\text{AdS}_{d+1}/\text{CFT}_d$ correspondence [49] in that all known extremal black holes contain the AdS_2 factor in their near horizon geometries.

It has been discussed in [50,51] that the motion of the particle near the horizon of the extreme Reissner–Nordström black hole is described by the (super)conformal mechanics. Since such black holes can be alternatively described by the wrapped M2-branes around a compact Riemann surface in M-theory, we expect that our superconformal quantum mechanics provides further examples and the M-theoretic interpretation.

It has been pointed out in [52,53] that the correlation functions of the DFF-model [12] have the expected scaling behaviors although one cannot assume the existence of the normalized and conformal invariant vacuum states in conformal quantum mechanics as in other higher-dimensional conformal field theories. We would like to extend the analysis to superconformal quantum mechanics including our models.

2. Indices and the reduced Gromov–Witten invariants

The formula for the numbers of genus g curves in a K3 surface, the so-called reduced Gromov–Witten invariants [54] has been firstly proposed by Yau and Zaslow in the analysis of the wrapped D3-brane [55]. Closely related to their setup, our $\mathcal{N} = 8$ superconformal gauged quantum mechanics (5.50) appears from the wrapped M2-branes instead of the D3-brane. It would be interesting to compute the indices and to extract enumerative information and structure from our model.

3. 1d–2d relation

In analogy with the fascinating stories arising from the compactification of M5-branes, for example, the AGT-relation [56], the DGG-relation [57] and the 2d–4d relation [58], it would be attractive to find the relationship between the superconformal field theories and the geometries or relevant dualities from M2-branes, i.e. “1d–2d relation”. It has been observed in [59] that the WDVV equation [60,61] and the twisted periods [62,63] which are relevant to two-dimensional geometries and topological field theories appear from the constraint conditions for the constructions of $\mathcal{N} = 4$ superconformal mechanics. It would be interesting to investigate whether our M-theoretical construction of superconformal quantum mechanics could help to understand and generalize such relations.

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